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Factorization of Fredholm Operators on Analytic Functions

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Let Ω be a simply connected domain in the complex plane, and $A(\Omega^n)$, the space of functions which are defined and analytic on Ω^n, if K is the operator on elements $u(t, a_1, \ldots, a_n)$ of $A(\Omega^{n+1})$ defined in terms of the kernels $k_i(t, s, a_1, \ldots, a_n)$ in $A(\Omega^{n+2})$ by $Ku = \sum_{i=1}^{n} \int_{a_i}^{t} k_i(t, s, a_1, \ldots, a_n)\, u(s, a_1, \ldots, a_n)\, ds \in A(\Omega^{n+1})$ and I is the identity operator on $A(\Omega^{n+1})$, then the operator $I - K$ may be factored in the form $(I - K)(M - W) = (I - \prod K)(M - \prod W)$. Here, W is an operator on $A(\Omega^{n+1})$ defined in terms of a kernel $w(t, s, a_1, \ldots, a_n)$ in $A(\Omega^{n+2})$ by $Wu = \int_{a_n}^{t} w(t, s, a_1, \ldots, a_n)\, u(s, a_1, \ldots, a_n)\, ds$. $\prod W$ is the operator; $\prod Wu = \int_{a_{n-1}}^{t} w(t, s, a_1, \ldots, a_n)\, u(s, a_1, \ldots, a_n)\, ds$. $\prod K$ is the operator; $\prod Ku = \sum_{i=1}^{n-1} \int_{a_i}^{t} k_i(t, s, a_1, \ldots, a_n)\, ds + \int_{a_{n-1}}^{t} k_n(t, s, a_1, \ldots, a_n)\, u(s, a_1, \ldots, a_n)\, ds$. The operator M is of the form $m(t, a_1, \ldots, a_n)I$, where $m \in A(\Omega^{n+1})$ and maps elements of $A(\Omega^{n+1})$ into itself by multiplication. The function m is uniquely derived from K in the following manner. The operator K defines an operator K^* on functions u in $A(\Omega^{n+2})$, by $K^*u = \sum_{i=1}^{n-1} \int_{a_i}^{t} k_i(t, s, a_1, \ldots, a_n)\, u(s, a_1, \ldots, a_{n+1})\, ds + \int_{a_{n+1}}^{t} k_n(t, s, a_1, \ldots, a_n)\, u((s, a_1, \ldots, a_{n+1})\, ds$. A determinant $\delta(I - K^*)$ of the operator $I - K^*$ is defined as an element $m^*(t, a_1, \ldots, a_{n+1})$ of $A(\Omega^{n+2})$. This is mapped into $A(\Omega^{n+1})$ by setting $a_{n+1} = t$ to give $m(t, a_1, \ldots, a_n)$. The operator $I - \prod K$ may be factored in similar fashion, giving rise to a chain factorization of $I - K$. In some cases all the matrix kernels k_i defining K are separable in the sense that $k_i(t, s, a_1, \ldots, a_n) = P_i(t, a_1, \ldots, a_n)\, Q_i(s, a_1, \ldots, a_n)$, where P_i is a $1 \times p_i$ matrix and Q_i is a $p_i \times 1$ matrix, each with elements in $A(\Omega^{n+1})$, explicit formulas are given for the kernels of the factors W. The various results are stated in a form allowing immediate extension to the vector-matrix case.

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1. Introduction

Let Ω denote a simply connected subdomain of the complex plane $\mathbb{C}$ and $A(\Omega^n)$ the space of functions which are defined and analytic on $\Omega \times \Omega \times \cdots \times \Omega$ (n times). This paper develops a theory concerned with the Volterra factorization of a class of Fredholm operators on $A(\Omega)$, the set of analytic functions in Ω, which is analogous to factorization theories on the real line developed in [1–3]. The most basic and obvious extension of Fredholm theory concerned with equations on the real line of the form

$$u(t) = g(t) + \int_a^b k(t, s)\, u(s)\, ds \tag{1.1}$$

considers kernels $k(t, s) \in A(\Omega^2)$, points $a, b \in \Omega$, and forcing functions $g(t) \in A(\Omega)$. Such kernels we call holomorphic, and these form an essential ingredient of the theory developed herein. However, this class of problem is somewhat restricted and does not include common kernels arising from the theory of ordinary differential equations. For example, the boundary value problem

$$\frac{dy}{dt}(t) = A(t)\, y(t), \qquad \lambda y(a) + \mu y(b) = \xi,$$

where ξ, $y(t)$ are n-vectors, $A(t)$, λ, μ are $n \times n$ matrices, $\lambda + \mu = I$, and the components of y and A are elements of $A(\Omega)$, can be transformed to an equivalent Fredholm equation

$$y(t) = \xi + \int_a^t \lambda A(s)\, y(s)\, ds + \int_b^t \mu A(s)\, y(s)\, ds$$

incompatible with form (1.1) since $\lambda + \mu \neq 0$. This suggests that a richer extension would be obtained by considering the theory of equations of the form

$$y(t) = g(t) + \int_a^t k_1(t, s)\, y(s)\, ds + \int_b^t k_2(t, s)\, y(s)\, ds,$$

where $k_1(t, s)$ and $k_2(t, s) \in A(\Omega^2)$, $g(t) \in A(\Omega)$, or more generally, equations of the form

$$y(t) = g(t) + \sum_{i=1}^{n} \int_{a_i}^{t} k_i(t, s)\, y(s)\, ds, \qquad k_i \in A(\Omega^2), \quad i = 1, 2, \ldots, n. \tag{1.2}$$

A special case of such an equation would arise as the multipoint analog of the example above, where the boundary condition is replaced by

$$\sum_{i=1}^{n} \lambda_i y(a_i) = \xi,$$

where λ_i are such that $\sum_{i=1}^{n} \lambda_i = 1$. The equivalent Fredholm equations can be shown to be

$$y(t) = \xi + \sum_{i=1}^{n} \int_{a_i}^{t} \lambda_i A(s)\, y(s)\, ds.$$

The fundamental elements of the theory developed here are operators of the form $K_a \equiv [k(t, s)]_a$, $k(t, s) \in A(\Omega^2)$; t, s, $a \in \Omega$, which map $A(\Omega)$ into itself in the following way: For any $u \in A(\Omega)$,

$$K_a u = w, \qquad \text{where} \quad w(t) = \int_a^t k(t, s)\, u(s)\, ds \in A(\Omega).$$

If we write I for the identity operator mapping each element u of $A(\Omega)$ into itself, the typical Fredholm problem (1.2) can be written

$$(I - K)u = g \qquad \text{where} \quad K = \sum_{i=1}^{n} K_{a_i} = \sum_{i=1}^{n} [k_i(t, s)]_{a_i}. \tag{1.3}$$

In the case $n = 1$, $I - K$ is a Volterra operator with an inverse of the same form; when $n = 2$ and $k_1(t, s) + k_2(t, s) = 0$, $I - K$ is a holomorphic Fredholm operator of the form (1.1). In general, K is called holomorphic if and only if

$$\sum_{i=1}^{n} k_i(t, s) = 0 \qquad \text{in} \quad A(\Omega^2). \tag{1.4}$$

In Section 2, the operators K and the function multiplication operators $\mathscr{M}$ whose action is defined as follows:

$$\begin{aligned} M &\equiv [m(t)] \in \mathscr{M} && \text{if} \quad m \in A(\Omega) \text{ and if for any } u \in A(\Omega), \\ Mu &\equiv [m(t)]u = w, && \text{where} \quad w(t) = m(t)\, u(t) \text{ for all } t \in \Omega, \end{aligned} \tag{1.5}$$

are shown to form subrings of the ring $\mathscr{O}$ of continuous linear operators on $A(\Omega)$. Multiplication in this ring is that induced by operator multiplication in $\mathscr{O}$, and it is shown that Fredholm operators of the form $M + K$ multiply according to the formulas

$$(M_1 + K_1)(M_2 + K_2) = M_3 + K_3,$$

where $M_i = [m_i(t)]$, $K_i = \sum_{j=1}^{n} [k_j{}^i(t, s)]_{a_j}$, and $i = (1, 2, 3)$ and

$$m_3(t) = m_1(t)\, m_2(t), \qquad k_l{}^3(t, s) = k_l{}^1(t, s)\, m_2(s) + m_1(t)\, k_l{}^2(t, s) + \omega_l(t, s), \tag{1.6}$$

and

$$\omega_l(t, s) = \sum_{j=1}^{n} \left(\int_{a_j}^{t} k_j^1(t, \theta)\, k_l^2(\theta, s)\, d\theta - \int_{a_l}^{s} k_l^1(t, \theta)\, k_j^2(\theta, s)\, d\theta \right).$$

In particular,

$$[h(t, s)]_{a_1}[k(t, s)]_{a_2} = \left[- \int_{a_1}^{s} h(t, \theta)\, k(\theta, s)\, d\theta \right]_{a_1} + \left[\int_{a_1}^{t} h(t, \theta)\, k(\theta, s)\, d\theta \right]_{a_2}.$$

For a given set of n points $\mathbf{a} = (a_1, a_2, ..., a_n)$, $a_i \in \Omega$, we denote by $\mathscr{I}(\mathbf{a})$ or $\mathscr{I}(\mathbf{a}, \Omega)$ the ring of operators K of form (1.3), by $\mathscr{F}(\mathbf{a})$ those like $M + K$, $M \in \mathscr{M}$, $K \in \mathscr{I}(\mathbf{a})$, and by $\mathscr{H}(\mathbf{a})$, the subring of holomorphic operators $K \in \mathscr{I}(\mathbf{a})$ which satisfy Eq. (1.4). $\mathscr{H}(\mathbf{a})$ is shown to be a two-sided ideal in $\mathscr{F}(\mathbf{a})$. Moreover, if $K \in \mathscr{I}(\mathbf{a})$ then $I - K$ may be uniquely factored in the form

$$I - K = (I - V)(I - H), \tag{1.7}$$

where $V \in \mathscr{I}(a_1)$: so that $I - V$ is an invertible Volterra operator, and $H \in \mathscr{H}(\mathbf{a})$.

The remainder of the paper is concerned with the problem of factoring $I - K$, $K \in \mathscr{I}(\mathbf{a})$ in the form

$$I - K = (I - V_1)(I - V_2) \cdots (I - V_n), \qquad V_i \in \mathscr{I}(a_i), \tag{1.8}$$

or something equivalent to it. Natural norms are defined on the elements $A(\Omega)$ and $\mathscr{I}(\mathbf{a})$, and it is shown that if K has sufficiently small norm then factorization of $I - K$ in form (1.8) is possible. Since the right-hand side of expression (1.8) is invertible such a form cannot be expected to hold in general. For example, consider Eq. (1.1) for the case $a = 0$ and b in Ω, $k(t, s) \equiv 1$ in Ω^2, so that in the notation of (1.3)

$$K = [1]_0 - [1]_b .$$

A factorization of form (1.8) is possible and in fact

$$\begin{aligned}(I - K) &= (I - V_1)(I - V_2) \\ &= (I - [1/(1 - s)]_0)(I + [1/(1 - t)]_b)\end{aligned} \tag{1.9}$$

provided Ω does not contain the point 1. Equation (1.9) can be interpreted as a factorization in terms of meromorphic functions in Ω^2, but in general this interpretation becomes untenable with three or more components in K. If Eq. (1.9) is rewritten in the form

$$(I - K)(I - V_2)^{-1} = (I - V_1),$$

that is,

$$(I - [1]_0 + [1]_b)(I - [1/(1 - s)]_b) = (I - [1/(1 - s)]_0),$$

then by multiplying both sides on the right by $[(1 - t)] \in \mathscr{M}$ we arrive at the factored form

$$(I - [1]_0 + [1]_a)([1 - t] - [1]_a) = ([1 - t] - [1]_0), \tag{1.10}$$

valid in the whole region Ω of analyticity of K. Now $1 - t$ may be characterized as the "Weiner–Hopf determinant" or "W–H determinant" of $I - K$. For an operator of the form

$$I - K = I - \sum_{i=1}^{n} [k_i(t, s, a_1, ..., a_n)]_{a_i}, \tag{1.11}$$

the W–H determinant, $D_n(t)$ of $I - K$ associated with a_n is defined as the determinant of the operator

$$I - K^* = I - \sum_{i=1}^{n-1} [k_i(t, s, a_1, ..., a_n)]_{a_i} - [k_n(t, s, a_1, ..., a_n)]_t .$$

Our main objective in this paper is to show that operator (1.11) may, after the manner of the example above, be factored in the same form

$$\begin{aligned}(I - K)([D_n(t)] - [u(t, s, a_1, ..., a_n)]_{a_n})\\ = (I - \bar{K})([D_n(t)] - [u(t, s, a_1, ..., a_n)]_{a_{n-1}}),\end{aligned} \tag{1.12}$$

where $\bar{K} \in \mathscr{I}(a_1, ..., a_{n-1})$, $u \in A(\Omega^{n+2})$, and $D_n(t) \in A(\Omega^{n+1})$. $(I - \bar{K})$ may be factored in a similar fashion and so on, thus building a chain of factorizations as the analog of (1.8). Specifically, $\bar{K} \in \mathscr{I}(a_1, a_2, ..., a_{n-1})$ is given by the following projection of K:

$$\bar{K} = \sum_{i=1}^{n-2} [k_i(t, s, a_1, ..., a_n)]_{a_i} + [k_{n-1}(t, s, a_1, ..., a_n) + k_n(t, s, a_1, ..., a_n)]_{a_{n-1}} . \tag{1.13}$$

An essential ingredient of the derivation of this general result is an embedding process in which we are led to consider families of operators K in which a_i are parameters. Since kernels derived as products using formula (1.6) are analytic functions of a_i we must consider the functions k_i as functions of $n + 2$ variables in $A(\Omega^{n+2})$ and $[k_i]_{a_i}$ as operators on $A(\Omega^{n+1})$. The ring product formulas (1.6) lead naturally to a kernel algebra $\mathscr{I}_n(\Omega)$ of elements $(k_1, k_2, ..., k_n)$ which we write alternatively as $\sum_{i=1}^{n} [k_i]_i$, where $k_i \equiv k_i(t, s, a_1, ..., a_n) \in A(\Omega^{n+2})$. This is isomorphic to the subring of $\mathscr{I}_n(\Omega_1)$, $\Omega_1 \subset \Omega$ which is obtained by considering elements of $\mathscr{I}_n(\Omega)$ restricted to Ω_1^{n+2} as elements of $\mathscr{I}_n(\Omega_1)$. If Ω_1 is small enough then any given $K \in \mathscr{I}_n(\Omega)$ will have its norm as an element of $\mathscr{I}_n(\Omega_1)$ sufficiently small to ensure the factorizability of $I - K$ in form (1.8) in Ω_1. In Section 6, we first define the determinant of a Volterra operator $(I - [v(t, s, a)]_a)$ as

$$\exp\left(-\int_a^t \operatorname{tr} v(\theta, \theta, \alpha)\, d\theta\right)$$

and then show that it is consistent to define the determinant of $I - K$ in Ω_1 as the product of the Volterra factors. This product in Ω_1^{n+1} has an analytic extension into Ω^{n+1} obtained by considering a class of degenerate kernels $\mathscr{D}(a_1, ..., a_n)$ which are dense in $\mathscr{H}(a_1, ..., a_n)$, and give an explicit representation for their determinants.

The operator $K \in \mathscr{H}(\mathbf{a})$ is degenerate if for some integer p,

$$k_i(t, s, \mathbf{a}) = \sum_{j=1}^{p} a_j(t, \mathbf{a})\, b_j^{\,i}(s, \mathbf{a}), \qquad t, s, \mathbf{a} \in \Omega^{n+2}, \tag{1.14}$$

where a_j, $b_j^{\,i} \in \Omega^{n+1}$ for $i = 1, 2, ..., n$; $j = 1, 2, ..., p$. More generally, if $K \in \mathscr{I}(\mathbf{a})$ and condition (1.14) is satisfied, then K is said to be separable. In Section 5, the basic factorization theorem is proved for separable kernels. The results are extended to general K in Section 7 as follows: The region Ω can be represented as the union of an infinite nested family of compact simply connected open sets Ω_r; such that $\Omega_{r_1} \subset \Omega_{r_2}$, $0 < r_1 \leqslant r_2 \leqslant 1$, $\Omega = \bigcup_{r<1} \Omega_r$. For any given r, $K \in \mathscr{H}(\mathbf{a})$ has a representation

$$K = D_r + N_r,$$

where $D_r \in \mathscr{D}(\mathbf{a}, \Omega)$ and N_r restricted to Ω_r has norm less than any given $\epsilon > 0$. We use this decomposition together with the results on factoring operators like $I - D_r$ and small norm operators like $I - N_r$ to obtain a factorization in Ω_r. The factorizations are analytic in Ω_r and unique in Ω_0 if Ω_0 is chosen small enough and therefore define a factorization in Ω. A factorization for general K follows from the result described by Eq. (1.7).

2. Notation and Preliminaries

Let Ω denote an open simply connected subset of the complex plane $\mathbb{C}$ and $A(\Omega^n)$ denote the space of functions which are defined and analytic on $\Omega \times \Omega \times \cdots \times \Omega$ (n times). If ω is an open subset with compact closure $\bar{\omega} \subset \Omega$ and $u \in A(\Omega^n)$, then $u \mid \omega^n$ denotes the restriction of u to ω^n and is a well-defined element of $A(\omega^n)$, and $|\, u\,|_\omega$ denotes the norm

$$|\, u\,|_\omega = \sup_{t_i \in \omega} |\, u(t_1, t_2, ..., t_n)|. \tag{2.1}$$

All operators considered in this paper are members of certain subrings of the ring $\mathcal{O}$ of continuous linear operators on $A(\Omega^n)$. The natural topology in this setting is that induced by the sup norm (2.1) over arbitrary open sets ω with compact closure in Ω.

Let $\mathcal{M}$ denote the ring associated with point-wise multiplication of function values on Ω defined as follows; For $m \in A(\Omega)$, the element $[m] \in \mathcal{M}$ is the operator taking elements of $A(\Omega)$ into $A(\Omega)$ according to

$$[m]u = v, \qquad \text{where} \quad v(t) = m(t)\, u(t), \qquad t \in \Omega. \tag{2.2}$$

For any $a \in \Omega$, $\mathcal{I}(a)$ denotes the set of operators $[k]_a$, $k \in A(\Omega^2)$ which send elements of $A(\Omega)$ into the same space according to the formula

$$[k]_a u = v, \qquad \text{where} \quad v(t) = \int_a^t k(t, s)\, u(s)\, ds, \qquad t \in \Omega. \tag{2.3}$$

The integral is taken over any path in Ω connecting a and t.

If $\mathbf{a} \equiv (a_1, a_2, \ldots, a_n)$ is a set of n points in Ω, then $\mathcal{I}(\mathbf{a})$ denotes the vector space of integral operators K of the form

$$K = \sum_{i=1}^{n} [k_i]_{a_i}, \qquad \text{where} \quad [k_i]_{a_i} \in \mathcal{I}(a_i), \qquad i = 1, 2, \ldots, n. \tag{2.4}$$

More generally, we denote by $\mathcal{F}(\mathbf{a})$, the vector space of Fredholm operators of the form $M + K$, where $M \in \mathcal{M}$ and $K \in \mathcal{I}(\mathbf{a})$.

Proposition 2.1. *$\mathcal{M}$ and $\mathcal{I}(\mathbf{a})$ are vector subspaces of $\mathcal{O}$.*

Proof. If $[m] \in \mathcal{M}$ and $[k]_a \in \mathcal{I}(a)$, let

$$v_n = [m]u_n \qquad \text{and} \qquad w_n = [k]_a u_n, \qquad \text{where} \quad u_n \in A(\Omega),\ n = 1, 2, \ldots .$$

We show that if u_n converges to zero as n tends to infinity (i.e., given any open set ω, ω compact in Ω, and any $\epsilon > 0$, there is an N such that for all $n_1, n_2 > N$, $| u_{n_1} - u_{n_2} |_\omega < \epsilon$), then v_n and w_n tend to zero as n tends to infinity.

Suppose ψ is a univalent mapping of the open unit disc Δ onto Ω. Denote by ω_r the range of ψ in Ω when its domain is restricted to the disc $\Delta_r = \{z : z \in \mathbb{C}, | z | < r < 1\}$. Since $d\psi/dz$ is bounded on the closure $\bar{\Delta}_r$ of Δ_r, the closure $\bar{\omega}_r$ of ω_r is compact and lies in Ω. Moreover, $\Omega = \bigcup_{r<1} \omega_r$, and every set ω with compact closure in Ω is contained in some ω_r. Thus, given any ω for which $\bar{\omega}$ has compact support in Ω, choose $r = r_1 < 1$ so that $\omega_{r_1} \supset \omega$. Then

$$| v_{n_1} - v_{n_2} |_\omega \leqslant | v_{n_1} - v_{n_2} |_{\omega_{r_1}} \leqslant | m |_{\omega_{r_1}} | u_{n_1} - u_{n_2} |_{\omega_{r_1}},$$

$$| w_{n_1} - w_{n_2} |_\omega \leqslant | w_{n_1} - w_{n_2} |_{\omega_{r_1}} \leqslant \sup_{t \in \omega_{r_1}} \left| \int_a^t k(t, s)\{u_{n_1} - u_{n_2}\}(s)\, ds \right|$$

$$\leqslant | \psi' |_{\Delta_{r_1}} \cdot | k |_{\omega_{r_1}} \cdot | u_{n_1} - u_{n_2} |_{\omega_{r_1}} \cdot 2r_1 .$$

Given $\epsilon > 0$, we can find N so that for all $n_1, n_2 > N$,

$$| u_{n_1} - u_{n_2} |_{\omega_{r_1}} < \epsilon \mid \psi' \mid_{\Delta_{r_1}} \cdot \mid k \mid_{\omega_{r_1}} \cdot 2r_1 \qquad \text{and} \qquad \epsilon / | m |_{\omega_{r_1}}.$$

Thus, both v_n and w_n tend to zero as n tends to infinity and hence $[m]$ and $[k]_a$ are continuous linear operators on $A(\Omega)$ and so belong to $\mathcal{O}$. Q.E.D.

PROPOSITION 2.2. *If* $M = [m] \in \mathcal{M}$ *and* $K = \sum_{i=1}^{n} [k_i]_{a_i} \in \mathcal{I}(\mathbf{a})$, *where* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $a_i \in \Omega$, and all a_i are distinct, then $M = K$ if and only if $m = 0$ and $k_i = 0$, $i = 1, 2, \ldots, n$.

Proof. We need only prove the "only if" part. Fix t, z in Ω and assume $t, z, a_1, a_2, \ldots, a_n$ are all distinct. Choose paths $\gamma_1, \gamma_1^*, \gamma_2, \ldots, \gamma_n$ in Ω such that:

(1) $(Ku)(t) = \sum_{i=2}^{n} \int_{\gamma_i} k_i(t, s)\, u(s)\, ds + \int_{\gamma_1} k_1(t, s)\, u(s)\, ds \equiv (K_1 u)(t) = \sum_{i=2}^{n} \int_{\gamma_i} k_i(t, s)\, u(s)\, ds + \int_{\gamma_1^*} k_1(t, s)\, u(s)\, ds \equiv (K_2 u)(t)$ for every $u \in A(\Omega)$;

(2) The compact sets $\Gamma_1 = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ and $\Gamma_2 = \gamma_1^* \cup \gamma_2 \cup \cdots \cup \gamma_n$ do not separate the plane;

(3) $\int_{\gamma_1 - \gamma_1^*} k_1(t, s)/(s - z)\, ds = 2\pi i k_1(t, z)$.

Let $L_i = M - K_i$, $i = 1, 2$, and observe that the operators L_1 and L_2 are continuous operators on $C(\Gamma_1)$ and $C(\Gamma_2)$, the spaces of continuous functions on Γ_1 and Γ_2. By hypothesis and construction, $L_i u = 0$ for every $u \in A(\Omega)$ and hence by Mergylan's theorem [5], $L_i f = 0$ for every $f \in C(\Gamma_i)$. Since $f(t) = 1/(t - z) \in C(\Gamma_i)$, $i = 1, 2$, we have

$$0 = L_1 f - L_2 f = \int_{\gamma_1 - \gamma_1^*} \frac{k_1(t, s)}{(s - z)}\, ds = 2\pi i k_1(t, z).$$

The arbitrariness of t, z implies $k_1 \equiv 0$ in Ω.

The same argument gives, successively, $k_i = 0$ for $i = 1, 2, \ldots, n$, and so $K = 0$. Thus $M = 0$ and this implies $M.\, 1 = m = 0$. Q.E.D.

COROLLARY 2.3. (1) *If* $K = \sum_{i=1}^{n} [k_i]_{a_i}$ *with* a_i*'s all distinct then* $K = 0$ *implies* $k_i = 0$, $i = 1, 2, \ldots, n$.

(2) *The identity operator* I *is not in* $\mathcal{I}(\mathbf{a})$ *for any* $\mathbf{a}$ *such that* $a_i \in \Omega$.

Proof. Items (1) and (2) come directly from Proposition 2.2 on setting $M = [0]$ and $M = [1]$, respectively. Q.E.D.

Corollary 2.3 shows that $\mathcal{I}(\mathbf{a})$ is isomorphic to the vector space of kernels $(k_1, k_2, \ldots, k_n)$, where $k_i \in A(\Omega^2)$.

It is a notational convenience to drop function arguments from expressions in cases where no confusion is likely to arise. In some places for the sake of brevity, we will write

$$\int_a^t hk, \int_a^t hu \qquad \text{instead of} \qquad \int_a^t h(t, \theta)\, k(\theta, s)\, d\theta, \int_a^t h(t, \theta)\, u(\theta)\, d\theta,$$

where $h, k \in A(\Omega^2)$ and $u \in A(\Omega)$.

PROPOSITION 2.4. *The vector spaces $\mathscr{M}$, $\mathscr{I}(\mathbf{a})$, $\mathscr{F}(\mathbf{a})$ are subrings of $\mathcal{O}$, where the multiplications in these subrings generated by the operator products in $\mathcal{O}$ are defined as follows: If $M_i = [m_i] \in \mathscr{M}$, $i = 1, 2, 3$, and $H = \sum_{i=1}^{n} [h_i]_{a_i}$, $K = \sum_{i=1}^{n} [k_i]_{a_i}$, $W = \sum_{i=1}^{n} [w_i]_{a_i} \in \mathscr{I}(\mathbf{a})$ then*

(1) *$M_1 M_2 = M_3$ if and only if $m_1 m_2 = m_3$,*

(2) *$HK = W$ if and only if $w_i(t, s) = \sum_{j=1}^{n} (\int_{a_j}^{t} h_j(t, \theta)\, k_i(\theta, s)\, d\theta - \int_{a_i}^{s} h_i(t, \theta)\, k_j(\theta, s)\, d\theta)$ for $i = 1, 2,\ldots, n$,*

(3) *$(M_1 + H)(M_2 + K) = (M_3 + W)$ if and only if $m_3 = m_1 m_2$, and*

$$w_i(t, s) = h_i(t, s)\, m_2(s) + m_1(t)\, k_i(t, s) + \sum_{j=1}^{n} \left(\int_{a_j}^{t} h_j(t, \theta)\, k_i(\theta, s)\, d\theta - \int_{a_i}^{s} h_i(t, \theta)\, k_j(\theta, s)\, d\theta\right) \tag{2.5}$$

Proof. We prove case (3), as (1) and (2) are consequences of it. For all $u \in A(\Omega)$,

$$\begin{aligned}
\{(M_1 + H)(M_2 + K)u\}(t) &= \left\{(M_1 + H)\left((m_2 u)(t) + \sum_{i=1}^{n} \int_{a_i}^{t} k_i(t, \theta)\, u(\theta)\, d\theta\right\}\right)(t) \\
&= m_1(t)\, m_2(t)\, u(t) + m_1(t) \sum_{i=1}^{n} \int_{a_i}^{t} k_i(t, \theta)\, u(\theta)\, d\theta \\
&\quad + \sum_{j=1}^{n} \int_{a_j}^{t} h_j(t, \theta)\, m_2(\theta)\, u(\theta)\, d\theta \\
&\quad + \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{a_j}^{t} h_j(t, \theta) \int_{a_i}^{\theta} k_i(\theta, s)\, u(s)\, ds\, d\theta \\
&= ([m_1 m_2]u)(t) \\
&\quad + \left(\sum_{i=1}^{n} [m_1(t)\, k_i(t, s) + h_i(t, s)\, m_2(s)]_{a_i} u\right)(t) \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} \left\{\int_{a_i}^{t} ds \int_{a_j}^{t} d\theta\, h_j(t, \theta)\, k_i(\theta, s)\, u(s) \right. \\
&\quad \left. - \int_{a_j}^{t} ds \int_{a_j}^{s} d\theta h_j(t, \theta)\, k_i(\theta, s)\, u(s)\right\} \\
&= \{(M_3 + W)u\}(t),
\end{aligned}$$

where $W = \sum_{i=1}^{n} [w_i]_{a_i}$ and

$$w_i(t, s) = m_1(t)\, k_i(t, s) + h_i(t, s)\, m_2(s) + \sum_{j=1}^{n} \left(\int_{a_j}^{t} h_j(t, \theta)\, k_i(\theta, s)\, d\theta - \int_{a_i}^{s} h_i(t, \theta)\, k_j(\theta, s)\, d\theta\right).$$

Thus the operator product of two elements of $\mathscr{F}(\mathbf{a})$ is equivalent to a unique third element (Corollary 2.3) of the same vector space and generates a ring structure in $\mathscr{F}(\mathbf{a})$. The vector spaces $\mathscr{M}$, $\mathscr{I}(\mathbf{a})$ are seen to be closed under this product and to form subrings of $\mathscr{F}(\mathbf{a})$. Q.E.D.

The rings $\mathscr{M}$, $\mathscr{I}(\mathbf{a})$, $\mathscr{F}(\mathbf{a})$ may be embedded in extended rings of operators by regarding $\mathbf{a}$ as a set of n parameters in Ω^n. Since kernels generated by the product formulas above are functions of $\mathbf{a}$ we must consider these extended operators as mapping $A(\Omega^{n+1})$ into itself. Denote these rings of operators on $A(\Omega^{n+1})$ by $\mathscr{M}_n$, $\mathscr{I}_n$, and $\mathscr{F}_n$. An element $K \in \mathscr{I}_n$ is essentially an ordered set of kernels $k_i(t, s, a) \in A(\Omega^{n+2})$, which we can write as

$$K = (k_1, k_2, \ldots, k_n) \qquad \text{or} \qquad K = \sum_{i=1}^{n} [k_i(t, s, \mathbf{a})]_i\,,$$

and an element M of $\mathscr{M}$ is of the form $M = [m(t, \mathbf{a})]$, where $m(t, \mathbf{a}) \in A(\Omega^{n+1})$. Products are given by the formulas of Proposition 2.4, and so the point evaluation map $\tau(\mathbf{a})$ projects $\mathscr{M}_n$, $\mathscr{I}_n$, and $\mathscr{F}_n$ homomorphically on the rings $\mathscr{M}$, $\mathscr{I}(\mathbf{a})$, and $\mathscr{F}(\mathbf{a})$. Since the n parameters $\mathbf{a}$ appear in all functions, elements of $A(\Omega^{n+1})$ and $A(\Omega^{n+2})$ will often be written in the form $m(t)$ and $k(t, s)$ for the sake of brevity.

Holomorphic operators play a special role in the theory and form a subring $\mathscr{H}(\mathbf{a})$ or $\mathscr{H}_n$ of the rings $\mathscr{I}(\mathbf{a})$ or $\mathscr{I}_n$, respectively.

Definition. If $K = \sum_{i=1}^{n} [k_i]_{a_i} \in \mathscr{I}(\mathbf{a})$, $k_i \in A(\Omega^2)$, and $\sum_{i=1}^{n} k_i(t, s) = 0$, then K is said to be holomorphic. Denote by $\mathscr{H}(\mathbf{a})$ the set of holomorphic operators in $\mathscr{I}(\mathbf{a})$. Evidently sums of operators in $\mathscr{H}(\mathbf{a})$ belong to $\mathscr{H}(\mathbf{a})$ and we have a subvector space of $\mathscr{I}(\mathbf{a})$. That the same is true for products, and hence $\mathscr{H}(\mathbf{a})$ is a subring, follows from the proposition:

Proposition 2.5. *$\mathscr{H}(\mathbf{a})$ is a two-sided ideal in $\mathscr{I}(\mathbf{a})$ and in $\mathscr{F}(\mathbf{a})$.*

Proof. Let $H = \sum_{i=1}^{n} [h_i]_{a_i} \in \mathscr{H}(\mathbf{a})$ and $K = \sum_{j=1}^{n} [k_j]_{a_j} \in \mathscr{I}(\mathbf{a})$. Consider the product

$$\begin{aligned} W_j = [k_j]_{a_j} H &= \sum_{i=1}^{n} [k_j]_{a_j} [h_i]_{a_i} \\ &= \sum_{i=1}^{n} \left\{ \left[-\int_{a_j}^{s} k_j(t, \theta)\, h_i(\theta, s)\, d\theta \right]_{a_j} + \left[\int_{a_j}^{t} k_j(t, \theta)\, h_i(\theta, s)\, d\theta \right]_{a_i} \right\} \\ &= \sum_{i=1}^{n} \left[\int_{a_j}^{t} k_j(t, \theta)\, h_i(\theta, s)\, d\theta \right]_{a_i}. \end{aligned}$$

This operator is in $\mathscr{H}(\mathbf{a})$ because $\sum_{i=1}^{n} \int_{a_j}^{t} k_j(t, \theta)\, h_i(\theta, s)\, d\theta = 0$, since $H \in \mathscr{H}(\mathbf{a})$, and in general $W = \sum_{j=1}^{n} W_j = KH$ belongs to $\mathscr{H}(\mathbf{a})$. Thus $\mathscr{H}(\mathbf{a})$ is a left ideal in $\mathscr{I}(\mathbf{a})$ and in fact in $\mathscr{F}(\mathbf{a})$ or even more generally in $\mathcal{O}$.

The right ideal property is demonstrated in a similar fashion. Let $U_j = H[k_j]_{a_j}$ and note that

$$U_j = \sum_{i=1}^{n} \left[-\int_{a_i}^{s} h_i(t, \theta)\, k_j(\theta, s)\, d\theta\right]_{a_i} + \left[\sum_{i=1}^{n} \int_{a_i}^{t} h_i(t, \theta)\, k_j(\theta, s)\, d\theta\right]_{a_j}$$

$$= \sum_{i=1}^{n} \left[-\int_{a_i}^{s} h_i(t, \theta)\, k_j(\theta, s)\, d\theta\right]_{a_i} + \left[\sum_{i=1}^{n} \int_{a_i}^{s} h_i(t, \theta)\, k_j(\theta, s)\, d\theta\right]_{a_j}.$$

Evidently $U_j \in \mathscr{H}(\mathbf{a})$ and so does $HK = \sum_{j=1}^{n} U_j$, so that H is a right ideal in $\mathscr{I}(\mathbf{a})$ and also in $F(\mathbf{a})$, since HM for any $M \in \mathscr{M}$ is obviously in $\mathscr{H}(\mathbf{a})$. Q.E.D.

The subring of operators $\mathscr{H}_n(\Omega)$ is defined in an analogous fashion. $K = \sum_{i=1}^{n} [k_i]_i \in \mathscr{H}_n(\Omega)$ if $K \in \mathscr{I}_n(\Omega)$ and $\sum_{i=1}^{n} k_i(t, s) = 0$ in $A(\Omega^{n+2})$. The ideal property also holds in this ring, since it is true for every projection by the point evaluation map $\tau(\mathbf{a})$ onto $\mathscr{F}(\mathbf{a})$.

If ψ is a given univalent map of the unit disc Δ with origin as center onto Ω and ψ_r is the image in Ω under ψ of the disc $\Delta_r \subset \Delta$ defined by

$$\Delta_r = \{z : z \in \mathbb{C},\ |\, z\,| < r < 1\},$$

then $\Omega = \bigcup_{r<1} \Omega_r$, and each Ω_r has compact closure in Ω.

Denote by $A_r(\Omega^{n+1})$, $n \geqslant 0$, the normed vector space with elements $u(t, \mathbf{a}) \in A(\Omega^{n+1})$ and norm $|\, u\,|_r = \sup_{t,\mathbf{a} \in \Omega_r^{n+1}} |\, u(t, a)| < \infty$. It is readily seen that $A_r(\Omega^{n+1}) \subsetneq A^\infty(\Omega_r^{n+1})$, the Banach space of bounded analytic functions in Ω_r^{n+1}, and that $A^\infty(\Omega_r^{n+1}) \subset A(\Omega_r^{n+1})$.

If $\mathbf{a} \in \Omega_r{}^n$ and $K = \sum_{i=1}^{n} [k_i]_{a_i} \in \mathscr{I}(\mathbf{a}, \Omega)$, then K may be considered as an element of the kernel algebra $\mathscr{I}_n(\Omega_r)$ by identifying it with the element of $\mathscr{I}_n(\Omega_r)$ with component kernels which are constant functions of $\mathbf{a}$ in Ω_r^{n+2}. Although this map does not preserve multiplication, it is useful as a technique for embedding individual elements of $\mathscr{I}(a, \Omega_r)$ into $\mathscr{I}_n(\Omega_r)$. Let $\tilde{K}_r$ denote the element of $\mathscr{I}_n(\Omega_r)$ identified by this embedding procedure.

Since Ω_r is compact we can define a norm on the elements of $\mathscr{I}(\Omega_r)$ as follows: For $K = \sum_{i=1}^{n} [k_i]_i \in \mathscr{I}_n(\Omega_r)$, define $|\, K\,|_r = (\sum_{i=1}^{n} |\, k_i\,|_r)\sigma_r$, where

$$\sigma_r = 2r \times |\, \psi'(\psi^{-1}(t))|_r\,. \tag{2.6}$$

The operator norm of an element K acting on $A^\infty(\Omega_r)$ denoted by $\|\, K\,\|_r$ is given by

$$\|\, K\,\|_r = \sup_{u \in A^\infty(\Omega_r)} (|\, Ku\,|_r / |\, u\,|_r), \tag{2.7}$$

and similarly the operator norm $\| K^* \|_r$ of $K^* \in \mathscr{I}_n(\Omega_r)$ is given by

$$\| K^* \|_r = \sup_{u \in A^\infty(\Omega_r^{n+1})} (| K^* u |_r / | u |_r). \tag{2.8}$$

PROPOSITION 2.6. *If* $K = \sum_{i=1}^n [k_i]_i \in \mathscr{I}_n(\Omega)$, *then*

$$\| K \|_r \leqslant | K |_r . \tag{2.9}$$

Moreover, if $H, K, W \in \mathscr{I}_n(\Omega)$ *and* $HK = W$, *then*

$$\| W \|_r \leqslant \| H \|_r \| K \|_r . \tag{2.10}$$

Proof. If $u \in A^\infty(\Omega_r^{n+1})$ and $w = Ku$, then

$$\begin{aligned} | w |_r &= \sup_{t, \mathbf{a} \in \Omega_r^{n+1}} \left| \sum_{i=1}^n \int_{a_i}^t k_i(t, \theta)\, u(\theta)\, d\theta \right| \\ &= \sup_{t^*, \mathbf{a}^* \in \Delta_r^{n+1}} \left| \sum_{i=1}^n \int_{a_i^*}^{t^*} k_i(\psi(t^*), \psi(\theta^*))\, u(\psi(\theta^*))\, \psi'(\theta^*)\, d\theta^* \right|, \end{aligned}$$

where $t^* = \psi^{-1}(t)$, $a_i^* = \psi^{-1}(a_i)$, and $\theta^* = \psi^{-1}(\theta)$, so that

$$| w |_r \leqslant \left(\sum_{i=1}^n | k_i |_r \right) \sigma_r \, | u |_r ,$$

and hence $\| K \|_r \leqslant | K |_r$.

Now suppose $H = \sum_{i=1}^n [h_i]_i$ and $W = \sum_{i=1}^n [w_i]_i$, and hence

$$w_i = \sum_{j=1}^n \left\{ \int_{a_j}^t h_j(t, \theta)\, k_i(\theta, s)\, d\theta - \int_{a_i}^s h_i(t, \theta)\, k_j(\theta, s)\, d\theta \right\}.$$

Then

$$\| w \|_r = \sup_{u \in A^\infty(\Omega_r^{n+1})} \left| \sum_{i=1}^n \int_{a_i}^t w_i(t, s)\, u(s)\, ds \right|_r \Big/ | u |_r .$$

Now,

$$\begin{aligned} &\sum_{i=1}^n \int_{a_i}^t \left\{ \sum_{j=1}^n \int_{a_j}^t h_j(t, \theta)\, k_i(\theta, s)\, d\theta - \sum_{j=1}^n \int_{a_i}^s h_i(t, \theta)\, k_j(\theta, s)\, d\theta \right\} u(s)\, ds \\ &\quad = \sum_{j=1}^n \int_{a_j}^t h_j(t, \theta) \left\{ \sum_{i=1}^n \int_{a_i}^\theta k_i(\theta, s)\, u(s)\, ds \right\} d\theta. \end{aligned}$$

Hence,

$$\| w \|_r = \sup_{u \in A^\infty(\Omega_r^{n+1})} \frac{| \sum_{j=1}^n \int_{a_j}^t h_j(t, \theta)\, v(\theta)\, d\theta |_r}{| v |_r} \frac{| v |_r}{| u |_r},$$

where

$$v(t) = \sum_{i=1}^n \int_{a_i}^t k_i(t, s)\, u(s)\, ds \in A^\infty(\Omega_r^{n+1}),$$

and so

$$\| w \|_r \leqslant \sup_{v \in A^\infty(\Omega_r^{n+1})} \frac{| \sum_{j=1}^n \int_{a_j}^t h_j(t, \theta)\, v(\theta)\, d\theta |_r}{| v |_r}$$

$$\times \sup_{u \in A^\infty(\Omega_r^{n+1})} \frac{| \sum_{i=1}^n \int_{a_i}^t k_i(t, s)\, u(s)\, ds |_r}{| u |_r}$$

$$\leqslant \| H \|_r \| K \|_r .$$ Q.E.D.

DEFINITION. An element $k(t, s, a) \in A(\Omega^{n+2})$ is separable if there are functions $f_1, ..., f_m$; $g_1, ..., g_m \in A(\Omega^{n+1})$ such that $f_1, ..., f_m$ are linearly independent and

$$k(t, s, \mathbf{a}) = \sum_{i=1}^m f_i(t, \mathbf{a})\, g_i(s, \mathbf{a}).$$

PROPOSITION 2.7. *If $k(t, s, \mathbf{a}) \in A(\Omega^{n+2})$, then for any $\epsilon > 0$ and $r < 1$ there exists a finite set of functions $f_1, ..., f_m$; $g_1, ..., g_m \in A(\Omega^{n+1})$ and a function $n \in A(\Omega^{n+2})$ such that*

$$k(t, s, \mathbf{a}) = \sum_{i=1}^m f_i(t, \mathbf{a})\, g_i(s, \mathbf{a}) + n(t, s, \mathbf{a}),$$

where $| n |_r < \epsilon$.

Proof. We may write

$$k(t, s, \mathbf{a}) = k(\psi(t^*), \psi(s^*), \psi(\mathbf{a}^*)) = h(t^*, s^*, \mathbf{a}^*),$$

where ψ is our previously chosen univalent mapping of Δ onto Ω. Now $h(t^*, s^*, a^*)$ is holomorphic in Δ^{n+2} and so has a power series

$$h(t^*, s^*, a^*) = \sum C_{p_1 p_2 \cdots p_{n+2}} a_1^{*p_1} a_2^{*p_2} \cdots a_n^{*p_n} t^{*p_{n+1}} s^{*p_{n+2}}$$

converging uniformly in Δ_r. For given $r < 1$, choose m so that

$$n^*(t^*, s^*, a^*) = h(t^*, s^*, a^*) - \sum_{0 \leqslant p_j \leqslant m} C_{p_1 \cdots p_{n+2}} a_1^{*p_1} \cdots a_n^{*p_n} t^{*p_{n+1}} s^{*p_{n+2}}$$

has norm $|n^*|_{\Delta_r} < \epsilon$. We now have

$$k(t, s, \mathbf{a}) = \sum_{0 \leqslant p_j \leqslant m} C_{p_1 \cdots p_{n+2}} (\psi^{-1}(a_1))^{p_1} \cdots (\psi^{-1}(a_n))^{p_n} (\psi^{-1}(t))^{p_{n+1}} (\psi^{-1}(s))^{p_{n+2}}$$
$$+ n(t, s, a),$$

where $n(t, s, a) = n^*(\psi^{-1}(t), \psi^{-1}(s), \psi^{-1}(a_1), ..., \psi^{-1}(a_n))$ and $|n|_r < \epsilon$. The polynomial may be factored in the form required. Q.E.D.

DEFINITION. Suppose $K = \sum_{i=1}^{n} [k_i]_i \in \mathscr{I}_n(\Omega)$ and each k_i is separable. Then the operator K is said to be separable. We define in a similar manner separable operators of $\mathscr{I}(\mathbf{a}, \Omega)$.

Suppose K is separable and $k_i(t, s, \mathbf{a}) = \sum_{j=1}^{m_i} f_j^i(t, \mathbf{a})\, g_j^i(s, \mathbf{a})$. If $a'(t, \mathbf{a})$ denotes the row vector $(f_1^1, f_2^1, ..., f_{m_1}^1, f_1^2, ..., f_{m_2}^2, ..., f_1^n, ..., f_{m_n}^n)$ and $b_i(s, \mathbf{a})$ the transpose of the row vector $(0, 0, ..., ..., g_1^i, g_2^i, ..., g_{m_i}^i, 0, ..., ..., 0)$, where g^i, follows $m_1 + m_2 + \cdots + m_{i-1}$ zeros, then

$$k_i(t, s, \mathbf{a}) = a'(t, \mathbf{a})\, b_i(s, \mathbf{a}), \tag{2.11}$$

where multiplication is the scalar product. We may derive from $a'(t, a)$ a vector composed of just its linearly independent elements and modify each b_i accordingly to obtain a new representation (2.11) in which a' is composed of linearly independent elements. The separable operator K may therefore be assumed to have a representation

$$K = \sum_{i=1}^{n} [a'(t, \mathbf{a})\, b_i(s, \mathbf{a})]_i\,, \tag{2.12}$$

where $a'(t, \mathbf{a})$ has linearly independent elements. Then if K is also holomorphic so that

$$a'(t, \mathbf{a}) \left\{ \sum_{i=1}^{n} b_i(s, \mathbf{a}) \right\} = 0,$$

then we must have

$$\sum_{i=1}^{n} b_i(s, \mathbf{a}) = 0. \tag{2.13}$$

DEFINITION. If $K \in \mathscr{H}_n(\Omega)$, ($\mathscr{H}(\mathbf{a}, \Omega)$) and is separable then K is said to be degenerate. The class of such operators is denoted by $\mathscr{D}_n(\Omega)$, ($\mathscr{D}(\mathbf{a}, \Omega)$).

PROPOSITION 2.8. *The elements $\mathscr{D}_n(\Omega)$ form an ideal in $\mathscr{I}_n(\Omega)$ and in $\mathscr{F}_n(\Omega)$.*

Proof. Let $K = \sum_{i=1}^{n} [k_i]_i \in \mathscr{I}_n(\Omega)$ and $D \in \mathscr{D}_n(\Omega)$. We may assume a representation (2.12) for $\mathscr{D}$ where (2.13) holds. If $U = KD = \sum_{i=1}^{n} [u_i]_i$, then

$$u_j(t, s, \mathbf{a}) = \sum_{i=1}^{n} \int_{a_i}^{t} k_i(t, \theta, \mathbf{a})\, a'(\theta, \mathbf{a})\, d\theta b_j(s, \mathbf{a})$$

$$= \bar{a}'(t, \mathbf{a})\, b_j(s, \mathbf{a}),$$

where $\bar{a}'(t, a) = \sum_{i=1}^{n} \int_{a_i}^{t} k_i(t, \theta, \mathbf{a})\, a'(\theta, \mathbf{a})\, d\theta$. Thus U is degenerate and also holomorphic from Eq. (2.13). On the other hand, if $W = DK$ then

$$w_i(t, s, \mathbf{a}) = a'(t, \mathbf{a})\, \bar{b}_i(s, \mathbf{a}),$$

where

$$\bar{b}_i(s, \mathbf{a}) = \sum_{i=1}^{n} \left\{ \int_{a_j}^{s} b_j(\theta, \mathbf{a})\, k_i(\theta, s, \mathbf{a})\, d\theta - \int_{a_i}^{s} b_i(\theta, \mathbf{a})\, k_j(\theta, s, \mathbf{a})\, d\theta \right\}.$$

Each w_i is separable and evidently $\sum_{i=1}^{n} \bar{b}_i(s, \mathbf{a}) = 0$, and so $U, W \in \mathscr{D}_n(\Omega)$. That the same property holds in $\mathscr{F}_n(\Omega)$ is a trivial extension.

PROPOSITION 2.9. *For any given* $K \in \mathscr{H}_n(\Omega)$, $r < 1$, and $\epsilon > 0$ we can find $D_r \in \mathscr{H}_n(\Omega)$ and $P_r \in \mathscr{H}_n(\Omega)$ such that

$$K = D_r + P_r$$

and $| P_r |_r < \epsilon$.

Proof. Let $K = \sum_{i=1}^{n} [k_i]_i$. By Proposition 2.7 we can find $n_i \in A(\Omega^{n+2})$ with $| n_i |_r < \epsilon/2n\sigma_r$ such that $k_i(t, s, \mathbf{a}) = \sum_{j=1}^{m_i} f_j^i(t, \mathbf{a})\, g_j^i(s, \mathbf{a}) + n_i(t, s, \mathbf{a})$, $i = 1, 2, \ldots, n-1$. σ_r is given by formula (2.6). Since $k_n = -\sum_{i=1}^{n-1} k_i$, define $n_n(t, s, \mathbf{a}) = -\sum_{i=1}^{n-1} n_i(t, s, \mathbf{a})$, so that $P \equiv \sum_{i=1}^{n} [n_i(t, s, a)]_i \in \mathscr{H}_n(\Omega)$ and $| P |_r = \sum_{i=1}^{n} | n_i |_r\, \sigma_r \leqslant (n-1)\epsilon/2n + (n-1)\epsilon/2n < \epsilon$. The separable parts of $k_i(t, s, \mathbf{a})$ generate a separable element $D \in \mathscr{I}_n(\Omega)$. But $K = D + N$ and K and N are holomorphic, and hence so is D. Q.E.D.

3. ALGEBRAIC PROPERTIES

In this section we consider some general algebraic properties of the rings $\mathscr{F}(\mathbf{a}, \Omega)$ and $\mathscr{F}_n(\Omega)$, starting with some basic isomorphisms.

If $\Omega_1 \subset \Omega$ then $A(\Omega^n) \subsetneqq A(\Omega_1{}^n)$, and so if $\mathbf{a}$ is such that $\mathbf{a}_i \subset \Omega_1$ for all i, then each element of $\mathscr{F}(a, \Omega)$ is also an element of $\mathscr{F}(\mathbf{a}, \Omega_1)$ so that $\mathscr{F}(\mathbf{a}, \Omega) \subsetneqq \mathscr{F}(\mathbf{a}, \Omega_1)$. In fact the ring $\mathscr{F}(\mathbf{a}, \Omega)$ is mapped isomorphically by this identification onto a subring of $\mathscr{F}(\mathbf{a}, \Omega_1)$.

Again, if an open simply connected region Ω^* is mapped by a univalent mapping function ψ onto Ω, and $\mathscr{M}^*$, $\mathscr{I}^*(\mathbf{a}^*)$, $\mathscr{F}^*(\mathbf{a}^*)$ denote the operator rings on $A(\Omega^*)$ derived from $\mathscr{M}$, $\mathscr{I}(\mathbf{a})$, $\mathscr{F}(\mathbf{a})$ on $A(\Omega)$, as follows:

$$[m(t)] \in \mathscr{M} \to [m(\psi(t^*))] \equiv [m^*(t^*)] \in \mathscr{M}^*,$$

$$K \equiv \sum_{i=1}^{n} [k_i(t, s)]_{a_i} \in \mathscr{I}(a, \Omega) \to K^* \equiv \sum_{i=1}^{n} [k(\psi(t^*), \psi(s^*))\psi'(s^*)]_{a_i^*} \in \mathscr{I}^*(\mathbf{a}^*, \Omega^*),$$

where $a_i = \psi(a_i^*)$, $t = \psi(t^*)$, $s = \psi(s^*)$, then this mapping is an isomorphism taking $\mathscr{F}(\mathbf{a}, \Omega)$ onto $\mathscr{F}(\mathbf{a}^*, \Omega^*)$.

If T denotes the map which takes $u \in A(\Omega)$ to $u^* = Tu \in A(\Omega^*)$, where $u^*(t^*) = u(\psi(t^*))$, and $[m] \in \mathscr{M}$, $K = \sum [k_i(t, s)]_{a_i} \in \mathscr{I}(\mathbf{a}, \Omega)$, then $T[m]T^{-1}$ is identified with $[m(\psi(t^*))] \in \mathscr{M}^*$ and TKT^{-1} with $K^* = \sum_{i=1}^{n} [k(\psi(t^*), \psi(s^*))\, \psi'(s^*)]_{a_i^*} \in \mathscr{I}^*(\mathbf{a}^*, \Omega^*)$, where $a_i = \psi(a^*)$.

Proposition 3.1. *If $M = [m(t) \in \mathscr{M}$, $[v]_a \in \mathscr{I}(a, \Omega)$ and $m(t) \neq 0$ for any $t \in \Omega$, then $M - [v]_a$ is invertible with a unique inverse $[m^{-1}(t)] + [w]_a$, $w \in A(\Omega^2)$, and*

$$\begin{aligned} w(t, s) - m^{-1}(t)\, v(t, s) &= \int_s^t m^{-1}(t)\, v(t, \theta)\, w(\theta, s)\, d\theta \\ &= \int_s^t w(t, \theta)\, m^{-1}(\theta)\, v(\theta, s)\, d\theta. \end{aligned} \tag{3.1}$$

Proof. By virtue of the isomorphism defined by ψ, mapping $\mathscr{F}(\mathbf{a}, \Omega)$ onto $\mathscr{F}(\mathbf{a}^*, \Delta^*)$ and vice versa, we need only prove the result on the unit disc. Consider first the case $M = [1]$. In this case Eq. (3.1) have a unique solution $w \in A(\Delta^2)$. This follows from the uniform convergence on compacta of Δ^2 of the series

$$w = \sum_{l=1}^{\infty} v_l\,, \quad v_1(t, s) = v(t, s),$$

$$v_l(t, s) = \int_s^t v_{l-1}(t, \theta)\, v(\theta, s)\, d\theta, \qquad l = 2, 3, \ldots$$

(see classical proof in [4]). If $m(t) \neq 0$ for any $t \in \Delta$, then $[m^{-1}] \in \mathscr{M}(\Delta)$ and $[m] - [v]_a = [m](I - [m^{-1}(t)\, v(t, s)]_a)$. The result now follows from the case $m = 1$. Q.E.D.

Proposition 3.2. *If $K = \sum_{i=1}^{n} [k_i]_i \in \mathscr{I}_n(\Omega)$, then $I - K$ may be uniquely factored in the form*

$$(I - K) = (I - [v]_1)(I - H),$$

where $H \in \mathscr{H}_n(\Omega)$.

Proof. We may write K in the form

$$K = \left[\sum_{i=1}^{n} k_i\right]_1 + \left(\sum_{i=1}^{n} [k_i]_i - \left[\sum_{i=1}^{n} k_i\right]_1\right) \equiv V + \bar{H}.$$

This decomposition into a "Volterra 1" element and a holomorphic component is unique, for suppose

$$[v_1]_1 + H^1 = [v_2]_1 + H^2, v_1, v_2 \in A(\Omega^{n+2}), \qquad H^1, H^2 \in \mathscr{H}_n(\Omega).$$

Then $[v_1]_1 - [v_2]_1 = [v_1 - v_2]_1 \in \mathscr{H}_n(\Omega)$ so that $v_1 - v_2 = 0$, $[v_1]_1 = [v_2]_1$, and hence $H^1 = H^2$. Let $I + W$ denote the inverse of $(I - [v]_1)$, where $v = \sum_{i=1}^{n} k_i$. Then

$$I - K = I - [v]_1 - \bar{H}, \qquad \bar{H} = \left(\sum_{i=1}^{n} [k_i]_i - \left[\sum_{i=1}^{n} k_i\right]_i\right)$$

$$= (I - [v]_1)(I - (I + W)\bar{H})$$

Since $(I + W)\bar{H} = H \in \mathscr{H}_n(\Omega)$. By proposition (2.5) we have

$$I - K = (I - [v]_1)(I - H).$$

This factorization is unique since

$$(I - [v_1]_1)(I - H^1) = (I - [v_2]_1)(I - H^2)$$

implies $[v_1]_1 + \tilde{H}^1 = [v_2]_1 + \tilde{H}^2$, where $\tilde{H}^i = (1 - [v_i]_1)H^i \in \mathscr{H}_n(\Omega)$, which as before implies $v_1 = v_2$ and $\tilde{H}^1 = \tilde{H}^2$ and so $H^1 = H^2$. Q.E.D.

DEFINITION. For $K \in \mathscr{I}(\mathbf{a}, \Omega)$, $I - K$ is proper factorizable if it can be expressed as a product of Volterra factors in the form

$$I - K = (I - [v_1]_{b_1})(I - [v_2]_{b_2}) \cdots (I - [v_p]_{b_p}), \tag{3.2}$$

where $v_i(t, s) \in A(\Omega^2)$ and $b_i = a_j$ for some $1 \leqslant j \leqslant n$, $i = 1, 2,..., p$.

PROPOSITION 3.3. *If $K \in \mathscr{I}(a, \Omega)$ and $I - K$ is proper factorizable in the form* (3.2) *and $b_i \neq b_j$ for $i \neq j$, then the factors are unique for the given ordering $b_1, b_2, ..., b_p$ of the components a_i of* $\mathbf{a}$.

Proof. Suppose $I - K$ has two proper factorizations of form (3.2), which we write as

$$(I - K) = (I - [v_1]_{b_1})(\cdots)(I - [v_p]_{b_p}) = (I - [u_1]_{b_1})(\cdots)(I - [u_p]_{b_p}).$$

By Proposition (3.1), $I - [v_i]_{b_i}$ and $I - [u_i]_{b_i}$ have unique inverses in the subring of $\mathscr{F}(\mathbf{a})$ generated by b_i components of $\mathscr{I}(\mathbf{a})$, and we can write

$$(I - [v_p]_{b_p})(I - [u_p]_{b_p})^{-1}$$
$$= (I - [v_{p-1}]_{b_{p-1}})^{-1} \cdots (I - [v_1]_{b_1})^{-1}(I - [u_1]_{b_1}) \cdots (I - [u_{p-1}]_{b_{p-1}}).$$

The left side and the right side are in subrings of $\mathscr{F}(\mathbf{a})$ which intersect only in $\mathscr{M}$ and so the left and right sides must equal I. Thus $v_p = u_p$ and similarly $v_i = u_i$ for $i = 1, 2,..., p$. Q.E.D.

Remarks. (1) If $I - K$ is proper factorizable, it is invertible.

(2) If $K \in \mathscr{I}_n(\Omega)$, then $I - K$ may be said to be proper factorizable if Eq. 3.2 holds, where $b_i = j$ for some $i \leqslant j \leqslant n$, $i = 1, 2,..., p$. The analog of Proposition 3.3 is proved in a similar manner.

We can define a factorization structure on the ring $\mathscr{F}_n(\Omega)$ (see [1] for definition) in terms of the projections $P_m{}^{\pm}$ defined as follows: Let $M + K$ be an arbitrary element of $\mathscr{F}_n(\Omega)$ where $M = [m]$ and $K = \sum_{i=1}^n [k_i]_i$. Then

$$\begin{aligned} P_m{}^{+}(M + K) &= M + \sum_{i=1}^{m-1} [k_i]_i , \\ P_m{}^{-}(M + K) &= M + \left[\sum_{i=m}^{n} k_i\right]_m . \end{aligned} \tag{3.3}$$

This structure is embedded in $\mathscr{F}_{n+1}(\Omega)$ by the additive map τ_m defined as follows:

$$\tau_m(M + K) = M + \sum_{i=1}^{m-1} [k_i]_i + \left[\sum_{i=m}^{n} k_i\right]_{n+1} . \tag{3.4}$$

The following proposition shows that τ_m is an embedding (see [1, Section 5]) of the factorization structure defined by $P_m{}^{\pm}$ in $\mathscr{F}_n(\Omega)$ into that in $\mathscr{F}_{n+1}(\Omega)$, defined by the projections $q_m{}^{\pm}$, where for $M + K^* \in F_{n+1}(\Omega)$, $K^* = \sum_{i=1}^{n+1} [k_i^*]_i$,

$$q_m{}^{+}(M + K^*) = M + \sum_{i=1}^{m-1} [k_i^*]_i , \qquad q_m{}^{-}(M + K^*) = M + \left[\sum_{i=m}^{n+1} k_i^*\right]_{n+1} .$$

PROPOSITION 3.4. *If*

$$U = \sum_{i=1}^{m-1} [u_i]_i , \qquad V = \sum_{i=1}^{n} [v_i]_i , \qquad W = \sum_{i=m}^{n} [w_i]_i \in \mathscr{I}_n(\Omega)$$

and τ_m *is the map from* $\mathscr{I}_n(\Omega)$ *to* $\mathscr{I}_{n+1}(\Omega)$ *defined by*

$$\tau_m K = \sum_{i=1}^{m-1} [k_i]_i + \left[\sum_{i=m}^{n} k_i\right]_{n+1} \in \mathscr{I}_{n+1}(\Omega), \qquad \text{for} \quad K = \sum_{i=1}^{n} [k_i]_i \in \mathscr{I}_n(\Omega),$$

then

$$\tau_m\{UVW\} = (\tau_m U)(\tau_m V)(\tau_m W) \qquad \text{in} \quad \mathscr{I}_{n+1}(\Omega). \tag{3.5}$$

Proof. The map τ_m is additive, and the result follows by observing that for $i < m$

$$\begin{aligned}\tau_m\{[u_i]_i[v_j]_j\} &= \left[-\int_{a_i}^{s} u_i(t, \theta)\, v_j(\theta, s)\, d\theta\right]_i + \left[\int_{a_i}^{t} u_i(t, \theta)\, v_j(\theta, s)\, d\theta\right]_{n+1} \\ &= (\tau_m[u_i]_i)(\tau_m[v_j]_j)\end{aligned}$$

and for $k \geqslant m$, $j \geqslant m$,

$$\tau_m\{[v_j]_j[w_k]_k\} = \left[\int_{s}^{t} v_j(t, \theta)\, w_k(\theta, s)\, d\theta\right]_{n+1} = (\tau_m[v_j]_j)(\tau_m[w_k]_k). \qquad \text{Q.E.D.}$$

COROLLARY. *If* U, V, W *are defined as above and*

$$(I - V)(I + W) = I - U,$$

then

$$(I - V)(I + W) = (I - \bar{V})(I + \bar{W}), \tag{3.6}$$

where

$$\bar{V} = \sum_{i=1}^{m-1} [v_i]_i + \left[\sum_{i=m}^{n} v_i\right]_{m-1}, \qquad \bar{W} = \left[\sum_{i=m}^{n} w_i\right]_{m-1}.$$

Proof. From Proposition 3.3 we see that

$$\begin{aligned}(I - V)(I + W) = I - U &= I - \tau_m U \\ &= (I - \tau_m V)(I + \tau_m W).\end{aligned}$$

The result follows by setting $a_{n+1} = a_{m-1}$. Q.E.D.

In the holomorphic ring of elements $M + K$ where $K \in \mathscr{K}_n(\Omega)$ we can define another factorization structure by means of the projections $\prod_m^{\pm}$ defined as follows: $\prod_m^{\pm} M = M$ for all $M \in \mathscr{M}$;

$$\prod_m^{+} K = \sum_{i=1}^{m-1} [k_i]_i + \left[\sum_{i=m}^{n} k_i\right],$$

$$\prod_m^{-} K = \left[\sum_{i=1}^{m} k_i\right]_m + \sum_{i=m+1}^{n} [k_i]_i \qquad \text{for} \quad K = \sum_{i=1}^{n} [k_i]_i \in \mathscr{K}_n(\Omega). \tag{3.7}$$

PROPOSITION 3.5. *If* $H = \sum_{i=1}^{n} [h_i]_i$, $K = \sum_{i=1}^{n} [k_i]_i \in \mathscr{H}_n(\Omega)$ *then*

$$\prod_m^+ \left\{ \left(\prod_m^+ H \right) K \right\} = \left(\prod_m^+ H \right) \left(\prod_m^+ K \right),$$
$$\prod_m^+ \left\{ H \left(\prod_m^- K \right) \right\} = 0. \tag{3.8}$$

Proof. The results follow by evaluating the expression on each side of the equations and comparing terms. Q.E.D.

This result enables us to factorize $1 - H$, $H \in \mathscr{H}_n(\Omega)$ in cases where $I - H_m{}^+ \equiv I - \prod_m^+ H$ is invertible in $\mathscr{H}_n(\Omega)$. Thus

$$I - H = I - H_m{}^+ - H_m{}^- = (I - H_m{}^+)(I - (I - H_m{}^+)^{-1} H_m{}^-).$$

Now $\prod_m^+ \{(I - H_m{}^+)^{-1} H_m{}^-\} = 0$ by Proposition 3.4 and so we have a factorization of $I - H$.

4. SMALL NORM THEORY

If $K \in \mathscr{I}_n(\Omega)$ it can be uniquely factored in the form

$$I - K = (I - [v]_1)(I - H),$$

where $H \in \mathscr{H}_n(\Omega)$. (see Proposition 3.2). Thus $I - K$ can be factored if and only if $I - H$ can. Moreover it can be shown that $| H |_r \leqslant [2 \mid K |_r/(1 - | K |_r)]$ so that the factorization of $I - K$ in the case of small norm can be approached either directly or via the small norm theory of factorization of holomorphic Fredholm operators.

PROPOSITION 4.1. *If* $K \in \mathscr{I}_n(\Omega)$ *and* $| K |_r < 1$, *then* $I - K$ *is invertible and*

$$(I - K)(I + W) = (I - U) \quad \text{and} \quad (I - K) = (I - [v_1]_1)(I - [v_2]_2) \cdots (I - [v_n]_n), \tag{4.1}$$

where W, U, *and* $[v_i]_i \in \mathscr{I}_n(\Omega_r)$, $W = [w_n]_n$, *and* $U = \sum_{i=1}^{n+1} [u_i]_i$.

Proof. The first of Eq. (4.1) requires w and u_i to satisfy equations

$$w_n(t, s) = k_n(t, s) + \sum_{j=1}^{n-1} \int_{a_j}^{t} k_j(t, \theta)\, w_n(\theta, s)\, d\theta + \int_s^t k_n(t, \theta)\, w_n(\theta, s)\, d\theta, \tag{4.2}$$

$$u_i(t, s) = k_i(t, s) - \int_{a_i}^{s} k_i(t, \theta)\, w_n(\theta, s)\, d\theta. \tag{4.3}$$

Equation (4.2) is of the form

$$w_n = k_n + K_s w_n ,$$

where K_s is a bounded linear operator on $A^\infty(\Omega_r^{n+2})$ with norm less than one. By classical theory there is a unique solution $w_n \in A^\infty(\Omega_r^{n+2})$ of (4.2) such that $| w_n |_r \leqslant | k_n |_r/(1 - \| K_s \|)$. Thus

$$\| W \|_r \leqslant \| K_s \|/(1 - \| K_s \|) \leqslant | K |_r/(1 - | K |_r). \tag{4.4}$$

Equation (4.3) gives u_i explicitly in terms of k_i and w for $i = 1, 2,..., n - 1$ and

$$| u_i |_r \leqslant | k_i |_r/(1 - \| K_s \|)$$

and so

$$\| U \|_r \leqslant \| K \|_r/(1 - \| K_s \|) \leqslant | K |_r/(1 - | K |_r). \tag{4.5}$$

The corollary to Proposition 3.3 gives

$$(I - K)(I + [w_n]_n) = (I - K_{n-1})(I + [w_n]_{n-1}),$$

where $K_{n-1} = \sum_{i=1}^{n-1} [k_i]_i + [k_n]_{n-1}$ for which $| K_{n-1} |_r \leqslant | K |_r < 1$ so that $I - K_{n-1}$ may be factorized in the ring $\mathscr{F}_n(\Omega_r)$ in the same way. This process may be continued as follows;

$$\begin{aligned} I - K &= (I - K_{n-1})(I + [w_n]_{n-1})(I + [w_n]_n)^{-1}, \\ I - K_{n-1} &= (I - K_{n-2})(I + [w_{n-1}]_{n-2})(I + [w_{n-1}]_{n-1})^{-1}, \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots \\ I - K_1 &= \left(I - \left[\sum_{i=1}^{n} k_i\right)_1\right] \end{aligned} \tag{4.6}$$

and by substitution and multiplication of factor pairs $(I + [w_{m-1}]_{m-1})^{-1}(I + [w_m]_{m-1})$, the required form of the second Eq. (4.1) is obtained. Finally, $I - K$ is invertible since all the Volterra factors are invertible. Q.E.D.

Remark. If the factors are multiplied in the pairs $(I + [w_m]_{m-1})(I + [w_m]_m)^{-1}$, a factorization of $I - K$ is obtained in terms of holomorphic factors of the form $(I + [h_m]_{m-1} - [h_m]_m)$ and a final Volterra factor $I - K_1$.

COROLLARY. *If $K \in \mathscr{I}_n(\Omega)$, then for any $r < 1$, $P_r \in \mathscr{I}_n(\Omega)$ and $D_r \in \mathscr{D}_n(\Omega)$ can be found such that $| P_r |_r < 1$ and*

$$I - K = (I - V_1)(I - P_r)(I - D_r). \tag{4.7}$$

Proof. By Proposition 3.2, $I - K$ may be uniquely factored in the form

$$(I - K) = (I - V_1)(I - H), \qquad \text{where} \quad H \in \mathscr{H}_n(\Omega).$$

By Proposition 2.9, given H and r we can find $D_r \in \mathscr{D}_n(\Omega)$ and $P_r \in \mathscr{H}_n(\Omega)$ such that $| P_r |_r < 1$ and $H = D_r + P_r$. Thus

$$I - H = I - P_r - D_r = (I - P_r)(I - (I - P_r)^{-1} D_r) = (I - P_r)(I - \bar{D}_r)$$

since $(I - P_r)^{-1}$ exists and is in $\mathscr{F}_n(\Omega)$ by Proposition 4.1. Finally $D_r \in \mathscr{H}_n(\Omega)$ because of the ideal property, Proposition 2.5. Q.E.D.

The following proposition is concerned with an important continuity aspect of factorization.

PROPOSITION 4.2. *If $K^l = \sum_{i=1}^n [k_i^l]_i \in \mathscr{I}_n(\Omega)$, $l = 1, 2, \ldots$, and K^l tends to $K = \sum_{i=1}^n [k_i]_i \in \mathscr{I}_n(\Omega)$ as l tends to infinity in the sense that for any $r < 1$, $| K - K^l |_r$ tends to zero as l tends to infinity, and if in $\mathscr{F}_n(\Omega)$,*

$$(I - K) = (I - [v_1]_1)(I - [v_2]_2) \cdots (I - [v_n]_n), \tag{4.8}$$

then for given $r < 1$ there is an L_r such that for all $l > L_r$

$$(I - K^l) = (I - [v_1^l]_1)(I - [v_2^l]_2) \cdots (I - [v_n^l]_n) \tag{4.9}$$

in $\mathscr{F}_n(\Omega_r)$, and v_i^l tends to v_i as l tends to infinity.

Proof. Write $I - K^l = I - K - M^l$, where $M^l = K^l - K = \sum_{i=1}^n [m_i^l]_i$ so that $\sum_{i=1}^n | m_i^l |_r$ tends to zero as l tends to infinity for any given $r < 1$. Now $I - K = (I - U)(I - [v_n]_n)$ from expression (4.8) and both $I - U$ and $(I - [v_n]_n)$ are invertible in $\mathscr{F}_n(\Omega)$. Let $(I - U)^{-1} = I + R$ and $(I - [v_n]_n)^{-1} = I + W_n$ so that

$$\begin{aligned} I - K^l &= (I - U)\{I - (I + R)\, M^l(I + W)\}(I - [v_n]_n) \\ &\equiv (I - U)(I - X^l)(I - [v_n]_n). \end{aligned}$$

Now given $r < 1$ we can choose L_r so that for all $l > L_r$

$$\| X^l \|_r \leqslant \|(I + R)\|_r \, \| M^l \|_r \, \| I + W \|_r = A_r \, \| M^l \|_r < 1$$

so that $I - X^l$ is factorizable. For $l > L_r$

$$(I - X_l) = (I - Y^l)(I - Z^l), \qquad Z^l = [z^l]_n, \qquad Y^l = \sum_{i=1}^{n-1} [y_i^l]_i$$

in $\mathscr{F}_n(\Omega_r)$. Equations (4.4) and (4.5) show $\| Y^l \|_r$ and $\| Z^l \|_r$ tend to zero as l tends to infinity. We now have

$$\begin{aligned} (I - K^l) &= \{(I - U)(I - Y^l)\}\{(I - Z^l)(I - [v_n]_n)\} \\ &\equiv (I - U^l)(I - [v_n^l]_n), \end{aligned}$$

where U^l tends to U and $[v_n^l]_n$ tends to $[v_n]_n$ as l tends to infinity.

Now U^l is of the form $\sum_{i=1}^{n-1} [u_i{}^l]_i$ and U^l tends to U in $\mathscr{I}_n(\Omega_r)$ as l tends to infinity and $I - U = (I - [v_1]_1)(\cdots)(I - [v_{n-1}]_{n-1})$ in $\mathscr{F}_{n-1}(\Omega_r)$. Given initially $r = r_0 < 1$ we first choose $1 > r_1 > r_0$ and derive as above

$$I - K^l = (I - U^l)(I - [v_n{}^l]_n) \qquad \text{in} \quad \mathscr{F}_n(\Omega_{r_1}), \qquad l > L_{r_1}.$$

Next U^l is factored in the same fashion in $\mathscr{F}_{n-1}(\Omega_{r_2})$, where $r_1 > r_2 > r_0$; and in an obvious inductive manner we obtain a complete factorization of $I - K^l$ for all $l > L_{r_n}$ in $\mathscr{F}_n(\Omega_{r_n}) r_1 > r_2 > \cdots > r_n > r_0$, with the required properties. Q.E.D.

5. Separable Kernels

In this section we investigate the factorability of $I - K$ where $K \in \mathscr{I}_n(\Omega)$ and is separable so that it has a representation (2.12). We consider first the case where $n = 1$ and $I - K$ is a Volterra operator.

Proposition 5.1. *The separable Volterra operator*

$$I - [v]_\alpha = I - [a'(t, \alpha)\, b(s, \alpha)]_\alpha, \qquad t, s, \alpha \in \Omega, \tag{5.1}$$

has its resolvent kernel w separable and given by

$$w(t, s, \alpha) = a'(t, \alpha)\, \Phi_c(t, s, \alpha)\, b(s, \alpha), \tag{5.2}$$

where $\Phi_c(t, s, \alpha) \in A(\Omega^3)$ and is the fundamental $N \times N$ matrix defined by

$$\Phi_c(t, s) = I + \int_s^t C(\theta, \alpha)\, \Phi_c(\theta, s)\, d\theta, \qquad C(\theta, \alpha) = b(\theta, \alpha)\, a'(\theta, \alpha). \tag{5.3}$$

Proof. The kernel w is immediately seen to satisfy the two resolvent equations

$$\begin{aligned} w(t, s, \alpha) &= v(t, s, \alpha) + \int_s^t v(t, \theta, \alpha)\, w(\theta, s, \alpha)\, d\theta \\ &= v(t, s, \alpha) + \int_s^t w(t, \theta, \alpha)\, v(\theta, s, \alpha)\, d\theta \end{aligned} \tag{5.4}$$

by virture of Eq. (5.3), and its adjoint equation

$$\Phi_c(t, s, \alpha) = I + \int_s^t \Phi_c(t, \theta, \alpha)\, C(\theta, \alpha)\, d\theta. \tag{5.5}$$

The matrix Φ_c is itself separable by virtue of the fundamental matrix property

$$\Phi_c(t, s, \alpha) = \Phi_c(t, \alpha, \alpha)\, \Phi_c(\alpha, s, \alpha)$$

and so $w = \bar{a}'(t, \alpha)\, \bar{b}(s, \alpha)$, where $\bar{a}'(t, \alpha) = a'(t, \alpha)\, \Phi_c(t, \alpha, \alpha)$ and $\bar{b}(s, \alpha) = \Phi_c(\alpha, s, \alpha)\, b(s, \alpha)$. The resolvent is unique by virtue of Proposition 3.1. Q.E.D.

PROPOSITION 5.2. *If $K = \sum [a'(t, \mathbf{a})\, b_i(s, \mathbf{a})]_i \in \mathscr{I}_n(\Omega)$, then $(I - K)$ may be factored in the form*

$$I - K = (I - V)(I - H), \tag{5.6}$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$,

$$V = [a'(t, \mathbf{a})\, B(s, \mathbf{a})]_1,$$

$$H = \sum_{i=1}^{n} [a'(t, \mathbf{a})\, \phi(t, a_1, \mathbf{a})\, b_i(s, \mathbf{a})]_i - [a'(t, \mathbf{a})\, \Phi(t, a_1, \mathbf{a})\, B(s, \mathbf{a})]_1,$$

and

$$B(s, \mathbf{a}) = \sum_{i=1}^{n} b_i(s, \mathbf{a}); \qquad \Phi(t, s, \mathbf{a}) = I + \int_s^t B(\theta, \mathbf{a})\, a'(\theta, \mathbf{a})\, \Phi(\theta, s, \mathbf{a})\, d\theta.$$

Proof. It is immediately verified by substitution that

$$V + H - VH = K.$$

The factored form (5.6) was shown in Proposition 3.2 to be unique. Q.E.D.

THEOREM 5.3. *If $H = \sum_{i=1}^{n} [a'(t)\, b_i(s)]_i \in \mathscr{D}_n(\Omega)$ so that $\sum_{i=1}^{n} b_i(s) = 0$, then $I - H$ can be factored in the following ways;*

$$(1)\quad (I - H)([d_n(t)] + w) = (I - H_{n-1})([d_n(t)] + \bar{w}) = [d_n(t)] + u, \tag{5.7}$$

where $w = [a'(t)\, R_n(s)\, b_n(s)]_n$, $r_p(s) = I - \sum_{i=1}^{p-1} \int_{a_i}^{s} b_i(\theta)\, a'(\theta)\, d\theta$, $R_p(s) =$ adjugate of $r_p(s)$, $d_p(t) =$ determinant of $r_p(t)$, $H_{n-1} = \sum_{i=1}^{n-1} [a'(t)\, b_i(s)]_i + [a'(t)\, b_n(s)]_{n-1}$, $\bar{w} = [a'(t)\, R_n(s)\, b_n(s)]_{n-1}$, *and* $u = \sum_{i=1}^{n-1} [u_i]_i + [0]_n$. *If d_n has no zeros in Ω^{n+1} then*

$$(I + w[d_n^{-1}])^{-1} = I - [a'(t)\, r_n^{-1}(t)\, b_n(s)]_n; \tag{5.8}$$

and

$$(2)\quad [D_m](I - H) = (I - H_m)([D_m] - \bar{H}_m),\ D_m = d_m(a_m), \tag{5.9}$$

where

$$H_m = \sum_{i=1}^{m} [a'(t)\, b_i(s)]_i + \left[a'(t) \sum_{i=m}^{n} b_i(s)\right]_m,$$

$$\bar{H}_m = \sum_{i=m}^{n} [a'(t)\, R_m(a_m)\, b_i(s)]_i - \left[a'(t)\, R_m(a_m) \sum_{i=m}^{n} b_i(s)\right]_m.$$

If $d_m(t)$ and $d_n(t)$ have no zeros in Ω^{n+1} then $I - \bar{H}_m[D_m^{-1}]$ is invertible and

$$(I - \bar{H}_m[D_m^{-1}])^{-1} = I + \sum_{i=m}^{n} [a'(t)\, r_n^{-1}(a_n)\, b_i(s)]_i - \left[a'(t)\, r_n^{-1}(a_n) \sum_{i=m}^{n} b_i(s)\right]_m$$

$$= I + \left[a'(t)\, r_n^{-1}(a_n) \sum_{i=1}^{m-1} b_i(s)\right]_m + \sum_{i=m}^{n} [a'(t)\, r_n^{-1}(a_n)\, b_i(s)]_i .$$

Proof. Choose a region $\Omega_1 \subset \Omega$ sufficiently small to ensure that $r_p(t) = I - \sum_{i=1}^{p-1} \int_{a_i}^{t} b_i(\theta)\, a'(\theta)\, d\theta$ is invertible for all $1 \leqslant p \leqslant n$ and a_i, t in Ω_1. Consider the following product in $\mathscr{F}_n(\Omega_1)$;

$$E_1 \equiv \left(I - \sum_{i=1}^{n} [a'(t)\, b_i(s)]_i)(I + [a'(t)\, r_n^{-1}(s)\, b_n(s)]_n\right)$$

$$= I - \sum_{i=1}^{n} [a'(t)\, b_i(s)]_i + [a'(t)\, r_n^{-1}(s)\, b_n(s)]_n + \sum_{i=1}^{n} \left[\int_{a_i}^{s} a'(t)\, b_i(\theta)\, a'(\theta)\, d\theta r_n^{-1}(s)\, b_n(s)\right]_i - \left[\sum_{i=1}^{n} \int_{a_i}^{t} a'(t)\, b_i(\theta)\, d\theta r_n^{-1}(s)\, b_n(s)\right]_n$$

$$= I + \sum_{i=1}^{n} \left[a'(t) \left\{-b_i(s) + \int_{a_i}^{s} b_i(\theta)\, a'(\theta)\, d\theta r_n^{-1}(s)\, b_n(s)\right]\right\}_i + \left[a'(t) \left\{I - \sum_{i=1}^{n} \int_{a_i}^{s} b_i(\theta)\, a'(\theta)\, d\theta\right\} r_n^{-1}(s)\, b_n(s)\right]_n ,$$

since $\sum_{i=1}^{n} \int_{s}^{t} b_i(\theta)\, a'(\theta)\, d\theta = 0$, because $H \in \mathscr{D}_n(\Omega)$, and so $\sum_{i-1}^{n} b_i(\theta) = 0$. Thus

$$E_1 = I + \sum_{i=1}^{n-1} \left[a'(t) \left\{\int_{a_i}^{s} b_i(\theta)\, a'(\theta)\, d\theta r_n^{-1}(s)\, b_n(s) - b_i(s)\right]\right\}_i + \left[a'(t) \left\{I - \sum_{i=1}^{n-1} \int_{a_i}^{s} b_i(\theta)\, a'(\theta)\, d\theta\right\} r_n^{-1}(s)\, b_n(s) - a'(t)\, b_n(s)\right]_n$$

$$= I - U^*, \quad \text{where} \quad U^* = \sum_{i=1}^{n-1} \left[a'(t) \left\{b_i(s) - \int_{a_i}^{s} b_i(\theta)\, a'(\theta)\, r_n^{-1}(s)\, b_n(s)\right\}\right]_i .$$

By the corollary to Proposition 3.4,

$$I - U^* = (I - \bar{H})(I + [a'(t)\, r_n^{-1}(s)\, b_n(s)]_{n-1}),$$

where $\bar{H} = \sum_{i=1}^{n-1} [a'(t)\, b_i(s)]_i + [a'(t)\, b_n(s)]_{n-1}$. If $d_n(t)$ is the determinant of the matrix $r_n(t)$, then in $\mathscr{F}_n(\Omega_1)$

$$\begin{aligned}
&(I - H)(I + [a'(t)\, r_n^{-1}(s)\, b_n(s)]_n)[d_n(t)] \\
&\quad = (1 - H)([d_n] + W) = (I - \bar{H})(I + [a'(t)\, r_n^{-1}(s)\, b_n(s)]_{n-1})[d_n(t)] \\
&\quad \therefore (I - H)([d_n] + W) = (I - \bar{H})([d_n] + \bar{W}) \\
&\quad = (I - U^*)[d_n(t)] = [d_n] + U.
\end{aligned}$$

Since all the elements of the last equation are in $\mathscr{F}_n(\Omega)$ and the equation holds in $\mathscr{F}_n(\Omega_1)$, it must hold by analytic continuation in $\mathscr{F}_n(\Omega)$.

If d_n has no zeros in Ω^{n+1} then

$$I + W[d_n^{-1}] = I + [a'(t)\, r_n^{-1}(s)\, b_n(s)]_n$$

and this is in $\mathscr{I}_n(\Omega)$ and by Proposition 3.1 is invertible. The inverse is $(I - [a'(t)\, r_n^{-1}(t)\, b_n(s)]_n)$ since

$$\begin{aligned}
E_2 &\equiv (I - [a'(t)\, r_n^{-1}(t)\, b_n(s)]_n)(I + [a'(t)\, r_n^{-1}(s)\, b_n(s)]_n) \\
&= I - [a'(t)\, r_n^{-1}(t)\, b_n(s) - a'(t)\, r_n^{-1}(s)\, b_n(s) \\
&\quad + \int_s^t a'(t)\, r_n^{-1}(t)\, b_n(\theta)\, a'(\theta)\, r_n^{-1}(s)\, b_n(s)]_n .
\end{aligned}$$

Now

$$\int_s^t b_n(\theta)\, a'(\theta)\, d\theta = -\sum_{i=1}^{n-1} \int_s^t b_i(\theta)\, a'(\theta)\, d\theta = \int_s^t \frac{dr_n}{d\theta}(\theta)\, d\theta = r_n(t) - r_n(s).$$

Thus,

$$E_2 = I - [a'(t)\{r_n^{-1}(t) - r_n^{-1}(s) + r_n^{-1}(t)(r_n(t) - r_n(s))\, r_n^{-1}(s)\}\, b_n(s)]_n = I.$$

The second part of the theorem is proved in an analogous fashion. In the operator ring $\mathscr{F}_n(\Omega_1)$, direct multiplication verifies the equation

$$I - H = (I - H_m)\left(I - \sum_{i=m}^{n} [a'(t)\, r_m^{-1}(a_m)\, b_i(s)]_i + \left[a'(t)\, r_m^{-1}(a_m) \sum_{i=m}^{n} b_i(s)\right]_m\right). \tag{5.11}$$

Equation (2) is obtained by multiplying both sides of Eq. (5.11) on the right by the constant element $[d_m(a_m)] \in \mathscr{M}$. All elements in Eq. (5.9) are then in $\mathscr{F}_n(\Omega)$ and the equation holds in $\mathscr{F}_n(\Omega_1)$. It therefore holds in $\mathscr{F}_n(\Omega)$. It is also readily verified that the expression given for $(I - \bar{H}_m)^{-1}$ indeed gives the identity when multiplied by $I - \bar{H}_m$. Q.E.D.

Remark. (1) We may factor $(1 - H_n)$ as a product of factors with two components by successive applications of formula (2). Thus

$$
\begin{aligned}
[D_{n-1}](I - H_n) &= (I - H_{n-1})([D_{n-1}] + [a'(t)\, R_{n-1} b_n(s)]_{n-1} \\
&\qquad - [a'(t)\, R_{n-1} b_n(s)]_n), \\
[D_{n-2}](I - H_{n-1}) &= (I - H_{n-2})([D_{n-2}] + [a'(t)\, R_{n-2} B_{n-1}(s)]_{n-2} \\
&\qquad - [a'(t)\, R_{n-2} B_{n-1}(s)]_{n-1}), \qquad (5.12) \\
&\cdots\cdots\cdots\cdots\cdots \\
[D_m][I - H_{m+1}) &= (I - H_m)([D_m] + [a'(t)\, R_m B_{m+1}(s)]_m \\
&\qquad - [a'(t)\, R_m B_{m+1}(s)]_{m+1}),
\end{aligned}
$$

where $R_m = R_m(a_m)$ and $B_m(s) = \sum_{i=m}^{n} b_i(s)$, and hence,

$$
\begin{aligned}
[D_2][D_3] \cdots [D_{n-1}](I - H) &= ([D_2] + F_2)([D_3] + F_3) \cdots (D_{n-1}] + F_{n-1}), \\
F_m &= [a'(t)\, R_m B_{m+1}(s)]_m - [a'(t)\, R_m B_{m+1}(s)]_{m+1}\,. \qquad (5.13)
\end{aligned}
$$

(2) If $\| H \|_r < 1$ then by Proposition 4.1, $(I - H)$ is factorizable in $\mathscr{F}_n(\Omega_r)$. Now Ω_r contains a region Ω_0 say in which $r_m(s)$ is invertible for all $1 \leqslant m \leqslant n$. In $\mathscr{F}_n(\Omega_0)$, the factors have a representation given by Theorem 5.3. Thus,

$$
\begin{aligned}
(I - H) &= (I - H_{n-1})(I + [a'(t)\, r_n^{-1}(s)\, b_n(s)]_{n-1})(I - [a'(t)\, r_n^{-1}(t)\, b_n(s)]_n), \\
(I - H_{n-1}) &= (I - H_{n-2})(I + [a'(t)\, r_{n-1}^{-1}(s)\, B_{n-1}(s)]_{n-2}) \\
&\qquad \times (I - [a'(t)\, r_{n-1}^{-1}(t)\, B_{n-1}(s)]_{n-1}), \\
&\cdots\cdots\cdots\cdots\cdots \qquad (5.14) \\
(I - H_m) &= (I - H_{m-1})(I + [a'(t)\, r_m^{-1}(s)\, B_m(s)]_{m-1})(I - [a'(t)\, r_m^{-1}(t)\, B_m(s)]_m), \\
&\cdots\cdots\cdots\cdots\cdots \\
(I - H_1) &= I,
\end{aligned}
$$

where $B_m(t) = \sum_{i=m}^{n} b_i(t)$. Now in $\mathscr{F}_n(\Omega_r)$

$$(I - H) = (I - [v_1]_1)(\cdots)(I - [v_n]_n) \qquad (5.15)$$

and factorization is unique, so that

$$v_n \equiv a'(t)\, r_n^{-1}(t)\, b_n(s) \qquad (5.16)$$

in Ω_0 and hence in Ω_r. Thus $a'(t)\, r(t_n^{-1})\, b_n(s) \in A(\Omega_r^{n+2})$. The inverse of $I - [V_n]_n$ is $I + [a'(t)\, r_n^{-1}(s)\, b_n(s)]_n$ in Ω_0 and so $a'(t)\, R_n^{-1}(s)\, b_n(s) \in A(\Omega_r^{n+2})$ also. By an analogous argument we have $a'(t)\, r_m^{-1}(t)\, B_m(s)$ and $a'(t)\, r_m^{-1}(s)\, B_m(s) \in A(\Omega_r^{n+2})$ and so in $\mathscr{F}_n(\Omega_r)$, $I - [V_m]_m$ has the representation

$$I - [V_m]_m = (I - [a'(t)\, r_m^{-1}(t)\, B_m(s)]_m)(I + [a'(t)\, r_{m+1}^{-1}(s)\, B_{m+1}(s)]_m) \tag{5.17}$$

for $m = 2, 3, \ldots, n-1$ and

$$I - [V_1]_1 = (I + [a'(t)\, r_2^{-1}(s)\, B_2(s)]_1). \tag{5.18}$$

(3) $\bar{H}_1[D_1^{-1}] = H$ since $H_1 = 0$ and the relationships which holds in small regions where D_1^{-1} exists, continues analytically into $\mathscr{D}_n(\Omega)$. We see from formula (5.10) that $(I - \bar{H}_m[D_m^{-1}])^{-1}$ is obtained as a simple projection from $(I - H)^{-1}$ which accumulates the first $m - 1$ kernels of $(I - H)^{-1}$ as an m kernel and leaves the rest of the elementary operators comprising it unchanged. If $(I - H)$ is invertible in $\mathscr{F}_n(\Omega)$ and this projection of its inverse is also invertible in the same ring then $I - \bar{H}_m[D_m^{-1}]$ can be constructed by means of a sequence of operations.

6. Determinants

We first define the determinant of a Volterra operator $I - [v]$. This is used to define the determinant of $I - K \in \mathscr{F}_n(\Omega)$ on any domains Ω_r such that $\| K \|_r < 1$ as the product of the determinants of its Volterra factors (see Proposition 4.1).

Finally, this function on Ω_r^{n+1} is shown to have a global extension to $A(\Omega^{n+1})$.

Definition. If $v(t, s, \mathbf{a}) \in A(\Omega^{n+2})$ then the determinant of $(I - [v]_i)$ or $\delta(I - [v]_i) \in A(\Omega^{n+1})$ is defined by

$$\delta(I - [v]_i) \equiv \exp\left(-\int_{a_i}^{t} \operatorname{tr} v(\theta, \theta, \mathbf{a})\, d\theta\right) \equiv \exp(-\operatorname{tr}[v]_i). \tag{6.1}$$

Some elementary properties follow immediately from this definition. If $(I - [u]_i)$ and $(I - [v]_i)$ are two Volterra elements of $\mathscr{F}_n(\Omega)$ in the subring of ith components, then

$$\begin{aligned} \delta\{(I - [u]_i)(I - [v]_i)\} &= \delta\left\{I - \left[u + v - \int_s^t uv\right]_i\right\} \\ &= \exp\left\{-\int_{a_i}^{t} \operatorname{tr}(u(\theta, \theta) + v(\theta, \theta))\, d\theta\right\} \\ &= \delta(I - [u]_i)\, \delta(I - [v]_i). \end{aligned} \tag{6.2}$$

Also, $\delta(I - [v]_i)$ depends continuously on v; for if $v_m = v + w_m$, where $| w_m |_r$ tends to zero as m tends to infinity, then for any $r < 1$,

$$\delta(I - [v_m]_i) - \delta(I - [v]_i)|_r = \left| \exp \left\{ -\int_{a_i}^{t} \operatorname{tr} w_m(\theta, \theta)\, d\theta \right\} - 1 \right|_r | \delta(I - [v]_i |_r ,$$

and this tends to zero as m tends to infinity.

Definition. If $K \in \mathscr{I}_n(\Omega)$ and $\| K \|_r < 1$, then in $\mathscr{F}_n(\Omega_r)$ we have (see Proposition 4.1)

$$I - K = (I - [v_1]_1)(I - [v_2]_2) \cdots (I - [v_n]_n)$$

and we define $\delta_r(I - K)$ the determinant of $I - K$ in $A(\Omega_r^{n+1})$ by the expression

$$\delta_r(I - K) \equiv \prod_{i=1}^{n} \delta(I - [v_i]_i) \in A(\Omega_r^{n+1}). \tag{6.3}$$

Multiplicative and extension properties of $\delta_r(1 - K)$ are now derived by studying separable kernels which are dense in $\mathscr{I}_n(\Omega)$ and by appealing to continuity and uniqueness properties of factors. First, a lemma on separable Volterra operators in $\mathscr{I}_n(\Omega)$ is given.

Lemma 6.1. *If $v(t, s) = a'(t)\, b(s) \in A(\Omega^{n+2})$, where $a(t)$ and $b(t)$ are column vectors of m components each in $A(\Omega^{n+1})$, and the components of $a(t)$ are linearly independent, then*

$$\delta(I - [v]_i) = \det(U^{-1}(t, a_i)) = \det(U(a_i , t)), \tag{6.4}$$

where the $m \times m$ matrix $U(t, s)$ is the fundamental matrix defined by

$$\begin{aligned} U(t, s) &= I + \int_s^t b(\theta)\, a'(\theta)\, U(\theta, s)\, d\theta \\ &= I + \int_s^t U(t, \theta)\, b(\theta)\, a'(\theta)\, d\theta. \end{aligned} \tag{6.5}$$

Proof. If $\Phi'(z) = A(z)\, \Phi(z)$ where Φ and A are $m \times m$ matrices then (see [6, p. 111]),

$$\det \Phi(z) = \det(\Phi(z_1)) \exp \int_{z_1}^{z} \operatorname{tr} A(\theta)\, d\theta.$$

Thus

$$\delta(I - [a'(t)\, b(s)]_i) = \exp\left\{-\int_{a_i} \operatorname{tr} a'(\theta)\, b(\theta)\, d\theta\right\}$$

$$= \exp\left\{-\int_{a_i}^{t} \operatorname{tr}(b(\theta)\, a'(\theta))\, d\theta\right\}$$

$$= \det(U(a_i\,,\, t)) = \det(U^{-1}(t, a_i)). \qquad \text{Q.E.D.}$$

PROPOSITION 6.2. *If $H = \sum_{i=1}^{n} [a'(t)\, b_i(s)]_i \in \mathscr{D}_n(\Omega)$, so that $\sum_{i=1}^{n} b_i(s) = 0$, and if $\| H \|_r < 1$ and H_r denotes H as an element of $\mathscr{D}_n(\Omega_r)$, then*

$$\delta(I - H_r) = \det\left(I - \sum_{i=1}^{n} \int_{a_i}^{a_n} b_i(\theta)\, a'(\theta)\, d\theta\right). \tag{6.6}$$

Proof. Since $\| H \|_r < 1$, we see by Theorem 5.3, second remark that in $\mathscr{F}_n(\Omega_r)$

$$(I - H) = (I - [v_1]_1)(I - [v_2]_2) \cdots (I - [v_n]_n),$$

where

$$I - [v_n]_n = I - [a'(t)\, r_n^{-1}(t)\, b_n(s)]_n\,,$$

$$I - [v_m]_m = (I - [a'(t)\, r_m^{-1}(t)\, B_m(s)]_m)(I + [a'(t)\, r_{m+1}^{-1}(s)\, B_{m+1}(s)]_m)$$

for $2 \leqslant m \leqslant n - 1$ and $v_1 = a'(t)\, r_2^{-1}(s)\, B_2(s)$, where

$$r_m(s) = I - \sum_{i=1}^{n-1} \int_{a_i}^{s} b_i(\theta)\, a'(\theta)\, d\theta, \qquad B_m(s) = \sum_{i=m}^{n} b_i(s).$$

Thus by Lemma 6.1

$$\delta(I - [a'(t)\, r_m^{-1}(t)\, B_m(s)]_m) = \det U_1{}^m(a_m\,,\, t),$$

where

$$U_1{}^m(t, s) = I + \int_s^t B_m(\theta)\, a'(\theta)\, r_m^{-1}(\theta)\, U_1{}^m(\theta, s)\, d\theta,$$

which has the solution $U_1{}^m(t, s) = r_m(t)\, r_m^{-1}(s)$.

Similarly,

$$\delta(I + [a'(t)\, r_m^{-1}(s)\, B_m(s)]_{m-1}) = \det(r_m^{-1}(a_{m-1})\, r_m(t))$$

and so

$$\delta(I - H_r) = \prod_{m=2}^{n} \det(r_m(a_m)\, r_m^{-1}(t)) \det(r_m^{-1}(a_{m-1})\, r_m(t))$$

$$= \det r_2^{-1}(a_1) \det r_n(a_n) = \det r_n(a_n),$$

since $r_m(a_{m-1}) = r_{m-1}(a_{m-1})$ and $r_2(a_1) = I$ so that $\det r_2^{-1}(a_1) = 1$. Thus

$$(I - H_r) = \det\left(I - \sum_{i=1}^{n} \int_{a_i}^{a_n} b_i(\theta)\, a'(\theta)\, d\theta\right). \qquad \text{Q.E.D.}$$

PROPOSITION 6.3. *If* $K = I - [k_1]_1 - [k_2]_2 \in \mathscr{F}_n(\Omega)$ *and*

$$I - K = (I - [u_1]_1)(I - [u_2]_2) = (I - [v_1]_2)(I - [v_2]_1),$$

then

$$\delta(I - [u_1]_1)\, \delta(I - [u_2]_2) = \delta(I - [v_1]_2)\, \delta(I - [v_2]_1). \tag{6.7}$$

Proof. First suppose K is separable and

$$k_i(t, s) = a'(t)\, b_i(s).$$

By Proposition 5.2 we can write

$$\begin{aligned} I - K &= (I - V_1)(I - H) \\ &= (I - [a'(t)\{b_1(s) + b_2(s)\}]_1) \\ &\quad \times (I + [a'(t)\, \Phi(t, a_1)\, b_2(s)]_1 - [a'(t)\, \Phi(t, a_1)\, b_2(s)]_2), \end{aligned}$$

where $\Phi(t, s) = I + \int_s^t (b_1(\theta) + b_2(\theta))\, a'(\theta)\, \Phi(\theta, s)\, d\theta$.

This may be factored further in accord with Theorem 4.1 in $\mathscr{F}_2(\Omega_{r_1})$ if r_1 is small enough that $|K|_{r_1} < 1$, to give

$$I - K = (I - V_1)(I - [d_1]_1)(I - [d_2]_2),$$

where, because of the uniqueness proposition, Proposition 3.3,

$$d_2 = u_2 \qquad \text{and} \qquad (I - V_1)(I - [d_1]_1) = I - [u_1]_1 .$$

Then in $\mathscr{F}_2(\Omega_{r_1})$,

$$\begin{aligned} E &\equiv \delta(I - u_1]_1)\, \delta(I - [u_2]_2) = \delta(I - V_1)\, \delta(I - [d_1]_1)\, \delta(I - [d_2]_2) \\ &= \det \Phi(a_1, t) \det\left(I + \int_{a_1}^{a_2} b_2(\theta)\, a'(\theta)\, \Phi(\theta, a_1)\, d\theta\right) \end{aligned}$$

by Lemma 6.1 and Proposition 6.2. Thus,

$$E = \det\left(\Phi(a_1, t) + \int_{a_1}^{a_2} b_2(\theta)\, a'(\theta)\, \Phi(\theta, t)\, d\theta\right).$$

On the other hand, if $I - K$ is factored in the form

$$\begin{aligned} I - K &= (I - V_2)(I - [e_2]_2)(I - [e_1]_1) = (I - [V_1]_2)(I - [V_2]_1) \\ &= (I - [a'(t)\{b_1(s) + b_2(s)\}]_2) \\ &\quad \times (I + [a'(t)\, \Phi(t, a_2)\, b_1(s)]_2 - [a'(t)\, \Phi(t, a_2)\, b_1(s)]_1), \end{aligned}$$

where $I - K \in \mathscr{F}_2(\Omega_{r_2})$, r_2 sufficiently small, then we have, as before,

$$\begin{aligned} \delta(I - [v_1]_2)\, \delta(I - [v_2]_1) &= \delta(I - V_2)\{\delta(I - [e_2]_2\, \delta(I - [e_1]_1)\} \\ &= \det\left(\Phi(a_2\,, t) - \int_{a_1}^{a_2} b_1(\theta)\, a'(\theta)\, \Phi(\theta, t)\, d\theta\right). \end{aligned}$$

But

$$\begin{aligned} &\det\left(\Phi(a_2\,, t) - \int_{a_1}^{a_2} b_1(\theta)\, a'(\theta)\, \Phi(\theta, t)\, d\theta\right) \\ &\quad = \det\left(\Phi(a_2\,, t) - \int_{a_1}^{a_2} (b_1(\theta) + b_2(\theta))\, a'(\theta)\, \Phi(\theta, t)\, d\theta \right. \\ &\qquad \left. + \int_{a_1}^{a_2} b_2(\theta)\, a'(\theta)\, \Phi(\theta, t)\, d\theta\right) \\ &\quad = \det\left(\Phi(a_1\,, t) + \int_{a_1}^{a_2} b_2(\theta)\, a'(\theta)\, \Phi(\theta, t)\, d\theta\right) \\ &\quad = E \qquad \text{in} \quad A(\Omega_r^{n+1}), \text{ where } r < r_1 \text{ and } r_2\,. \end{aligned}$$

By analytic continuation, we therefore have

$$\delta(I - [u_1]_1)\, \delta(I - [u_2]_2) = \delta(I - [v_1]_2)\, \delta(I - [v_2]_1).$$

In general, for a given $K \in \mathscr{I}_2(\Omega)$ we can construct a sequence of separable kernels $K^m \in \mathscr{I}_2(\Omega)$ such that K^m tends to K as m tends to infinity, since by Proposition 2.9 we can choose a sequence r_n tending to one and ϵ_n tending to zero and find D_n, P_n such that

$$K = D_n + P_n\,, \qquad D_n \in \mathscr{D}_n(\Omega), \qquad \| P_n \|_{r_n} < \epsilon_n\,.$$

Set $K^m = D_m$. Then, by Proposition 4.2, for any given $r < 1$ there is an M such that for $m > M$

$$(I - K^m) = (I - [u_1{}^m]_1)(I - [u_2{}^m]_2) = (I - [v_1{}^m]_2)(I - [v_2{}^m]_1)$$

in $\mathscr{F}_2(\Omega_r)$, and $u_i{}^m$ tends to u_i, $v_i{}^m$ tends to v_i as m tends to infinity. Since

$$\delta(I - [u_1{}^m]_1)\, \delta(I - [u_2{}^m]_2) = \delta(I - [v_1{}^m]_2)\, \delta(I - [v_2{}^m]_1)$$

and $\delta(I - [v]_i)$ depends continuously on v, result (6.7) must follow. Q.E.D.

COROLLARY. *If* $K \in \mathscr{I}_n(\Omega)$ *and*

$$\begin{aligned} I - K &= (I - [u_1]_1)(I - [u_2]_2) \cdots (I - [u_n]_n) \\ &= (I - [v_1]_{m_1})(I - [v_2]_{m_2}) \cdots (I - [v_p]_{m_p}), \end{aligned} \tag{6.8}$$

where $1 \leqslant m_i \leqslant n$, *then*

$$\delta^* = \prod_{i=1}^{n} \delta(I - [v_i]_i) = \prod_{j=1}^{p} \delta(I - [v_j]_{m_j}). \tag{6.9}$$

Proof. Consider the sequence of integers $m_1, m_2, \ldots, m_p$. Locate the members which equal one. Either they all cluster at the start of the sequence or there is a first $m_q = 1$ with numbers in the sequence greater than one before it. In this second case, multiply the two factors $(I - [v_{q-1}]_{m_{q-1}})$ and $(I - [v_q]_{m_q})$ together and refactor in the opposite order. We obtain a new factorization of $(I - K)$ valid in some $\mathscr{F}_n(\Omega_1)$ with m_q moved a step toward m_1. This operation preserves δ^* in $A(\Omega_1^{n+1})$ and $I - K$ in $\mathscr{F}_n(\Omega_1)$. If m_q still has larger numbers before it, repeat the process until m_q takes the place of m_1 or joins the cluster of ones at $m_1, \ldots$. Next repeat the process for the next $m_q i = 1$ until all ones in the sequence are clustered at the start; then multiply all these together to give a factor $[I - (v_1^*)]_1$ as the first factor of a sequence in which $m_r > 1$ for $r > 1$. This process may be repeated to bring together next all m_r factors equal to two and so on. In a finite number of steps a new factorization of $I - K$ is produced of the same form as the first of (6.8) valid in some $\mathscr{F}_n(\Omega_t)$, $\Omega_t \subset \Omega_{t-1} \subset \cdots \subset \Omega_1$. At every step the value of δ^* is preserved in Ω_t^{n+1}. But the final factor form must be identical with

$$(I - [u_1]_1)(I - [u_2]_2) \cdots (I - [u_n]_n) \qquad \text{(see Proposition 3.3).}$$

Thus Eq. (6.9) holds in $A(\Omega_t^{n+1})$ and, therefore, by analytic continuation in $A(\Omega^{n+1})$. Q.E.D.

For an arbitrary $K = \sum_{i=1}^{n} [k_i]_i \in \mathscr{I}_n(\Omega)$, define $\delta(I - K) \in A(\Omega^{n+1})$ as follows:

DEFINITION. Given t, $\mathbf{a}$ in Ω^{n+1}, choose $r < 1$ so that $t, \mathbf{a} \in \Omega_r^{n+1}$. By the corollary to Proposition 4.1 we can factor $(I - K)$ in $\mathscr{F}_n(\Omega_r)$ in the form

$$I - K = (I - V_1)(I - P_r)(I - D_r), \tag{6.10}$$

where

$$V_1 = \left[\sum_{i=1}^{n} k_i\right]_1, \qquad \| P_r \|_r < 1, \qquad \text{and} \qquad D_r = \sum_{i=1}^{n} [a'(t)\, b_i(s)]_i \in \mathscr{D}_n(\Omega).$$

Define

$$\delta(I - K) = \delta(I - V_1)\, \delta_r(I - P_r) \det\left(I - \sum_{i=1}^{n-1} \int_{a_i}^{a_n} b_i(\theta)\, a'(\theta)\, d\theta\right). \quad (6.11)$$

PROPOSITION 6.4. *The determinant* $\delta(I - K)$ *is well defined and belongs to* $A(\Omega^{n+1})$.

Proof. Since $\| P_r \| < 1$, $I - P$ is factorizable in $\mathscr{F}_n(\Omega_r)$ and $\delta_r(I - P_r) \in A(\Omega_r^{n+1})$. The other two factors of (6.10) are in the same set and so

$$G_r = \delta(I - V_1)\, \delta_r(I - P_r) \det\left(I - \sum_{i=1}^{n-1} \int_{a_i}^{a_n} b_i(\theta)\, a'(\theta)\, d\theta\right) \in A(\Omega_r^{n+1}).$$

Choose r_0 small enough to ensure that $\| K \|_{r_0}$ and $\| D_r \|_{r_0}$ are both less than one. Then by Propositions 6.2 and 6.3 and corollary

$$G_r \mid \Omega_0^{n+1} = \delta_{r_0}(I - K),$$

where $\delta_{r_0}(I - K)$ is the product of the determinants of all the Volterra factors of $(I - K)$ in $\mathscr{F}_n(\Omega_{r_0})$. Thus G_r is an analytic continuation of $\delta_{r_0}(I - K) \in A(\Omega_{r_0}^{n+1})$ and so is independent of the decomposition (6.10). If $1 > r_2 > r_1$, then $G_{r_2} \mid \Omega_{r_1}^{n+1} = G_{r_1}$ so that $\delta(I - K)$ is well defined and belongs to $A(\Omega^{n+1})$.
Q.E.D.

Remark. If $K \in \mathscr{H}_n(\Omega)$, then $V_1 = 0$ in expression (6.10). $\delta(I - D)$ is independent of t and since $\| P \|_r$ can be chosen as small as one likes, $| \delta_r(I - P)|_r$ can be taken as near one as desired and so $\delta(I - K)$ must be independent of t, i.e., $\delta(I - H) \in A(\Omega^n)$ for $H \in \mathscr{H}_n(\Omega)$.

PROPOSITION 6.5. *If* $K_1, K_2, K_3, \in \mathscr{I}_n(\Omega)$ *and*

$$I - K_3 = (I - K_1)(I - K_2), \quad (6.12)$$

then

$$\delta(I - K_3) = \delta(I - K_1)\, \delta(I - K_2). \quad (6.13)$$

Proof. Both sides of Eq. (6.13) are elements of $A(\Omega^{n+1})$. Choose r such that $\| K_i \|_r < 1$ for $i = 1, 2$, and 3, so that

$$(I - K_i) = (I - [v_1{}^i]_1) \cdots (I - [v_n{}^i]_n), \qquad v_j{}^i \in A(\Omega^{n+2}), \quad i = 1, 2, 3,$$

in $\mathscr{F}_n(\Omega_r)$, and

$$\delta(I - K_i) \mid \Omega_r^{n+1} = \delta_r(I - [v_1{}^i]_1) \times \cdots \times \delta_r(I - [v_n{}^i]_n).$$

Equation (6.12) implies

$$(I-[v_1{}^3]_1)(I-[v_2{}^3]_2)\cdots((I-[v_n{}^3]_n)$$
$$=(I-[v_1{}^1]_1)\cdots(I-[v_n{}^1]_n)(I-[v_1{}^2]_1)\cdots(I-[v_n{}^2]_n)$$

and by the corollary to Proposition 6.3 that Eq. (6.13) holds in $A(\Omega_r^{n+1})$. It thus holds in $A(\Omega^{n+1})$ by analytic continuation. Q.E.D.

PROPOSITION 6.6. *If $H\in\mathscr{H}_n(\Omega)$, then $I-H$ is invertible in $\mathscr{F}_n(\Omega)$, if and only if $\delta(I-H)$ is invertible in $A(\Omega^n)$, that is, if and only if $\delta(I-H)\neq 0$ at any point $\mathbf{a}\in\Omega^n$.*

Proof. If $I-H$ is invertible in $\mathscr{F}_n(\Omega)$ and $I+Q$ is its inverse (and hence $Q\in\mathscr{H}_n(\Omega)$ by the ideal property), then

$$(I-H)(I+Q)=I; \qquad \delta(I-H)\,\delta(I+Q)=\delta I=1, \tag{6.14}$$

and so $\delta(I-H)$ is invertible.

Conversely, for any $r<1$ we can find P_r, D_r such that

$$(I-H)=(I-P_r)(I-D_r); \qquad D_r\in\mathscr{D}_n(\Omega_n), \quad P_r\in\mathscr{H}_n(\Omega_n), \quad \|P_r\|_r<1.$$

$I-P$ is factorizable and if

$$I-P_r=(I-[v_1]_1)(I-[v_2]_2)\cdots(I-[v_n]_n)\in\mathscr{H}_n(\Omega),$$

then $\delta(I-P_r)=\exp\{-\sum_{i=1}^n\int_{a_i}^{a_n}\operatorname{tr}v_i(\theta,\theta)\,d\theta\}$ and is invertible in $A(\Omega_r{}^n)$. Since $\delta_r(I-H)=\delta_r(I-P_r)\,\delta_r(I-D_r)$, then $\delta(I-H)$ invertible implies $\delta_r(I-D_r)$ invertible in $A(\Omega_r{}^n)$. From Proposition 6.2, if $D_r=\sum_{i=1}^n[a'(t)\,b_i(s)]_i$ and $r_n(a_n)=(I-\sum_{i=1}^{n-1}\int_{a_i}^{a_n}b_i(\theta)\,a'(\theta)\,d\theta)$, then $\{\delta(I-D_r)\}^{-1}=\det r_n(a_n)\in A(\Omega_r{}^n)$ and is invertible in $A(\Omega_r{}^n)$. Thus $r_n^{-1}(a_n)$ exists in $A(\Omega_r{}^n)$ and from formula (5.10) we see $I-\bar{H}_m$ is invertible in $\mathscr{F}_n(\Omega_r)$, and $\bar{H}_1\equiv H$.

The invertibility of $\delta(I-H)$ thus implies that of $I-H$ in every $\mathscr{F}_n(\Omega_r)$, $r<1$, and hence in $\mathscr{F}_n(\Omega)$. For if $I+Q_r=I-H)^{-1}$ in $\mathscr{F}_n(\Omega_r)$ and $r_1>r_2$, then $Q_{r_1}\mid\Omega_{r_2}^{n+2}=Q_{r_2}$ since the inverse is unique in $\mathscr{F}_n(\Omega_{r_2})$. Thus Q defined equal to Q_r for points in Ω_r^{n+2} satisfies

$$(I-H)(I+Q)=(I+Q)(I-H)=I. \qquad \text{Q.E.D.}$$

If $K\in\mathscr{I}(\mathbf{a},\Omega)$, the determinant of $I-K$ is defined as follows: Let $\tilde{K}=\sum_{i=1}^n[k_i]_i\in\mathscr{I}_n(\Omega)$ be such that $k_i(t,s)$ is independent of $\mathbf{a}$ for all i and identical with the corresponding kernels in K. Then $\delta(I-\tilde{K})$ evaluated at $\mathbf{a}$ is the determinant $\delta(I-K)\in A(\Omega)$ of $I-K$.

It can be shown after the manner of the proof of Proposition 6.6 that if $K\in\mathscr{H}(\mathbf{a},\Omega)$ then $I-K$ is invertible in $\mathscr{F}(\mathbf{a},\Omega)$ if and only if $\delta(I-K)\neq 0$.

Note that if $K \in \mathscr{I}(\mathbf{a}, \Omega)$, we can write $I - K = (I - V_1)(I - H)$ as in Proposition 3.2. $I - V_1$ is always invertible and so the invertibility of $I - K$ depends on $\delta(I - H) \in \mathbb{C}$.

7. Global Factorization

The globally defined determinant of an operator $I - K \in \mathscr{F}_n(\Omega)$ enables us to extend Theorem 5.3 to arbitrary elements of $\mathscr{F}_n(\Omega)$. The key step is a characterization of $d_m(t)$ in the general setting.

From Proposition 6.2 we see that

$$\begin{aligned} d_m(t) &= \det \left\{ I - \sum_{i=1}^{m-1} \int_{a_i}^{t} b_i(\theta)\, a'(\theta)\, d\theta \right\} \\ &= \delta(I - H_m^*), \end{aligned} \tag{7.1}$$

where

$$H_m^* = I - \sum_{i=1}^{m-1} [a'(t, \mathbf{a})\, b_i(s, \mathbf{a})]_i - [a'(t, \mathbf{a})\, B_m(s, \mathbf{a})]_t \tag{7.2}$$

and $B_m(s, \mathbf{a}) = \sum_{i=m}^{n} b_i(s, \mathbf{a})$. The a_i dependence of the vectors a' and b_i has been stressed to make it clear that (7.2) is not obtained from $I - H$ merely by substituting t for $a_m, a_{m+1}, \ldots, a_n$. These paramenters must be left untouched in the kernels of the operator. This can be done in the following way:

Elements $K = \sum_{i=1}^{n} [k_i]_i \in \mathscr{I}_n(\Omega)$ may be mapped into $\mathscr{I}_{n+1}(\Omega)$ by the map τ_m defined below by its action on K.

$$\tau_m K = \sum_{i=1}^{m-1} [k_i]_i + \left[\sum_{i=m}^{n} k_i \right]_{n+1} = \tilde{K}_m \in \mathscr{I}_{n+1}(\Omega). \tag{7.3}$$

Note that if $H = \sum_{i=1}^{n} [a'(t)\, b_i(s)]_i \in \mathscr{D}_n(\Omega)$, then

$$d_m(I - H)(a_{n+1}) = \delta(I - T_m H).$$

Definition. The Weiner–Hopf determinant, $d_m(I-K)$ of $I-K$, $K \in \mathscr{I}_n(\Omega)$, is defined as follows:

$$d_m(I - K)(t) = \delta(I - \tau_m K)(a_{n+1} = t)$$

and is an element of $A(\Omega^{n+1})$.

The map τ_m is not a homomorphism, but it does preserve certain products in the ring $\mathscr{F}_n(\Omega)$ (see Proposition 3.4).

THEOREM 7.1. *If $H = \sum_{i=1}^{n} [h_i]_i \in \mathscr{H}_n(\Omega)$, then $I - H$ may be factored in $\mathscr{F}_n(\Omega)$ as follows:*

$$(I - H)([d_n(I - H)] + [w]_n) = (I - H_{n-1})([d_n(I - H)] + [w]_{n-1}), \tag{7.5}$$

where $w \in A(\Omega^{n+2})$, $H_{n-1} = \sum_{i=1}^{n-1} [h_i]_i + [h_n]_{n-1}$;

$$[\delta(I - H_m)](I - H) = (I - H_m)([\delta(I - H_m)] - \bar{H}_m), \tag{7.6}$$

where $H_m = \sum_{i=1}^{m-1} [h_i]_i + [\sum_{i=m}^{n} h_i]_m$ and $\bar{H}_m = \sum_{i=m}^{n} [h_i]_i \in \mathscr{H}_n(\Omega)$.

Proof. By Proposition 2.9, for any $H \in \mathscr{H}_n(\Omega)$, and any $r < 1$ we can find $D_r \in \mathscr{D}_n(\Omega)$ and $P_r \in \mathscr{H}_n(\Omega_r)$ such that

$$H = P_r + D_r \qquad \text{and} \qquad |P_r|_r < 1.$$

Then $I - H = I - P_r - D_r$ and since $I - P_r$ may be factored in the form

$$(I - P_r)(I + [q]_n) = (I - \bar{P}_r)(I + [q]_{n-1})$$

and all factors are invertible in $\mathscr{F}_n(\Omega_r)$, we may write

$$(I - H)(I + [q]_n) = (I - \bar{P}_r)(I + [q]_{n-1})(I - D_r^*),$$

where

$$D_r^* = (I + [q]_{n-1})^{-1}(I - \bar{P}_r)^{-1} D_r(I + [q]_n) \in \mathscr{D}_n(\Omega_r).$$

By Theorem 5.3

$$(I - D_r^*)([d_n(I - D_r^*)] + [e]_n) = (I - \bar{D}_r^*)([d_n(I - D_r^*)] + [e]_{n-1}),$$

where if we write $D_r^* = \sum_{i=1}^{n} [d_i^*]_i$ then $\bar{D}_r^* = \sum_{i=1}^{n-1} [d_i^*]_i + [d_n^*]_{n-1}$. From Proposition 3.4 it can be seen that

$$\begin{aligned} d_n(I - D_r^*) &= \delta(\tau_m\{(I + [q]_{n-1})^{-1}(I - \bar{P}_r)^{-1}(I - H)(I + [q]_n)\}) \\ &= d_n\{(I + [q]_{n-1})^{-1}(I - \bar{P}_r)^{-1}(I + [q]_n)\}\, d_n(I - H) \\ &= d_n(I - P_r)^{-1}\, d_n(I - H) \end{aligned}$$

and hence in $\mathscr{F}_n(\Omega_r)$ we have

$$\begin{aligned} &(I - H)(I + [q]_n)([d_n(I - D_r^*)] + [e]_n)[d_n(I - \bar{P}_r)] \\ &\quad = (I - H)([d_n(I - H)] + [w_r]_n) = (I - \bar{H})([d_n(I - H)] + [w_r]_{n-1}), \end{aligned} \tag{7.7}$$

where $w_r \in A(\Omega_r^{n+2})$.

We show that w_r coincides with a unique element $w \in A(\Omega^{n+2})$ in Ω_r. Suppose w_{r_1} and w_{r_2} are constructed as above for $1 > r_1 > r_2 > r_0$. If r_0 is small enough, $I - \tau_n H$ is invertible in $\mathscr{F}_{n+1}(\Omega_{r_0})$ and so $[d_n(I - H)]$ is invertible in $A(\Omega_{r_0}^{n+2})$. In this case $(I + [w_{r_i}]_n[d_n(I - H)]^{-1})^{-1}$ is a factor of $I - H \in \mathscr{F}_n(\Omega_{r_0})$ and hence unique so that w_{r_1} and w_{r_2} coincide in $\Omega_{r_0}^{n+2}$ and so w_{r_1} is an analytic continuation of w_{r_2}. We define $w \in A(\Omega^{n+2})$ as that analytic function which coincides with w_r in Ω_r^{n+2}. It follows by the previous argument that w is well defined, unique, and satisfies Eq. (7.5).

The second part, Eq. (7.6) of the theorem is proved in an analogous manner. First we prove the small norm case.

If $H \in \mathscr{H}_n(\Omega)$ and $| H |_r < 1$, then in $\mathscr{F}_n(\Omega_r)$, $I - H_m$ is invertible and its inverse $I + Q$ is of the form $I + \sum_{i=1}^{m-1} [q_i]_i - [\sum_{i=1}^{m-1} q_i]_m$. Now

$$I - H = I - \prod_m^{+} H - \prod_m^{-} H$$

$$= (I - H_m)\left(I - (I + Q) \prod_m^{-} H\right) = (I - H_m)(I - P),$$

where, from Proposition 3.5, $\prod_m^{+} P = 0$ and so P has the form $\sum_{i=m}^{n} [P_i]_i \in \mathscr{H}_n(\Omega_r)$, and so $(I - H) = (I - H_m)(I - P)$. These factors, $(I - H_m)$ and $(I - P)$, are unique. This follows from [1] since a factorization structure is invoked, or by the elementary argument used earlier (Proposition 3.3). They are also invertible since $(I - H)$ and $(I + Q)$ are invertible in $\mathscr{F}_n(\Omega_r)$.

Proceeding as in the proof of Eq. (7.5), we write

$$\begin{aligned} I - H &= I - P^r - D^r \\ &= (I - P_m{}^r)(I - Q_m) - D^r, \end{aligned}$$

where $Q_m = \sum_{i=m}^{n} [\bar{q}_i]_i \in \mathscr{H}_n(\Omega_r)$ and so

$$I - H = (I - P_m{}^r)(I - D^{*r})(I - Q_m),$$

where $D^{*r} = (I - P_m{}^r)^{-1} D^r(I - Q_m)^{-1} \in \mathscr{D}_n(\Omega_r)$. $I - D^{*r}$ is factored in the manner of Theorem 5.3 (2), so that

$$\begin{aligned} &[d_m(I - D^{*r})(a_m)](1 - H) \\ &\quad = (I - P_m{}^r)(I - D_m^{*r})([d_m(I - D^{*r})(a_m)] - \bar{D}_m^{*r})(I - Q_m). \end{aligned} \tag{7.8}$$

Again from Proposition 3.5

$$\begin{aligned} d_m(I - D^{*r})(a_m) &= d_m\{(I - P_m{}^r)^{-1}(I - H)(I - Q_m)^{-1}\}(a_m) \\ &= d_m(I - H)\, d_m(I - P^r)(a_m) \end{aligned}$$

and by multiplying both sides of (7.8) by $d_m(I - P^r)^{-1}(a_m)$ we get

$$[d_m(I - H)(a_m)](I - H) = (I - H_m)([d_m(I - H)(a_m)] - R_m),$$

where $R_m = \sum_{i=m}^{n} [r_i]_i \in \mathscr{H}_n(\Omega_r)$, since from Proposition 3.5; by applying $\prod_m^+$ to Eq. (7.8), we get

$$(I - H_m) = (I - P_m{}^r)(I - D_m^{*r}).$$

Now $d_m(I - H)(a_m) = \delta \prod_m^+ (I - H) = \delta(I - H_m)$. An analytic continuation argument after the manner of the first part completes the proof. Q.E.D.

Remark. (1) If $K \in \mathscr{I}_n(\Omega)$ we may factor $I - K$ by first writing $I - K = (I - V_1)(I - H)$, $H \in \mathscr{H}_n(\Omega)$, and applying Theorem 7.1. Since $d_n(I - V_1) = \delta(I - V_1) \in A(\Omega^{n+1})$ and $(I - V_1)(I - H_{n-1}) = I - K_{n-1}$, where for $K = \sum_{i=1}^{n} (k_i)_i \in \mathscr{I}_n(\Omega)$, $K_{n-1} = \sum_{i=1}^{n-1} [k_i]_i + [k_n]_{n-1}$, we can write the result of Theorem 7.1 in the more general form

$$(I - K)([d_n(I - K)] + [w]_n) = (I - K_{n-1})([d_n(I - K)] + [w]_{n-1})$$

for $K \in \mathscr{I}_n(\Omega)$ and some $w \in A(\Omega^{n+2})$.

(2) Equation (7.6) may also be reformulated in the forms:

$$(I - H)([\delta(I - H)] + Q) = (I - H_m)[\delta(I - H)]$$

for $H \in \mathscr{H}_n(\Omega)$ and some $Q \in A(\Omega^{n+2})$ satisfying $\prod_m^+ Q = 0$;

$$(I - K)([\delta(I - K)] + \bar{Q}) = (I - K_m)[\delta(I - K)]$$

for $K \in \mathscr{I}_n(\Omega)$ and some $\bar{Q} \in A(\Omega^{n+2})$ satisfying $\prod_m^+ \bar{Q} = 0$.

In conclusion it should be noted that all proofs are readily translated to the matrix case where the elements of $A(\Omega^{n+1})$ are vectors and those of $A(\Omega^{n+2})$ are matrices.

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